

# Pseudo-Frobenius graded algebras with enough idempotents

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Dedicated to Alberto Facchini in his 60th birthday

## Abstract

We introduce and study the notion of pseudo-Frobenius graded algebra with enough idempotents, showing that it follows the pattern of the classical concept of pseudo-Frobenius (PF) and Quasi-Frobenius (QF) ring, in particular finite dimensional self-injective algebras, as studied by Nakayama, Morita, Faith, Tachikawa, etc. We show that such an algebra is characterized by the existence of a graded Nakayama form. Moreover, we prove that the pseudo-Frobenius property is preserved and reflected by covering functors, a fact which makes the concept useful in Representation Theory.

**Keywords:** Pseudo-Frobenius algebra, self-injective algebra, Nakayama automorphism, Nakayama form, covering functor.

Classification Code: 16Gxx ; Representation theory of rings and algebras.

## 1 Introduction

In Ring Theory and Representation Theory of Algebras it has been a classical and natural process to abstract the properties of group algebras over a field and define new contexts where many of those properties still hold and to which, as a consequence, results from group algebras are easily transferred. In this vein, the concept of finite dimensional self-injective algebra already appeared in the work of Nakayama ([14], [15]) and is nowadays a common object of study. On the ring-theoretic side, Quasi-Frobenius (QF) rings are the 'field-free' equivalent of finite dimensional self-injective algebras and were intensively studied, among others, by Tachikawa (see [19]). The most important feature of a Quasi-Frobenius ring is that its (left or right) module category is a Frobenius category, i.e., the classes of injective and projective modules coincide. Going further in this direction, Osofsky [17] introduced a class of rings, later called pseudo-Frobenius (PF). Left PF rings are those rings  $R$  for which the regular module  ${}_R R$  is an injective cogenerator. Using known characterizations of such rings (see [5][Theorem 24.32]), it is not hard to prove that these are precisely the rings for which the classes of finitely generated projective and finitely cogenerated injective left modules coincide.

When dealing with split basic finite dimensional algebras over a field  $K$ , one of the most important tools in Representation Theory is that of coverings of quivers with relations and, more

generally, covering functors of small  $K$ -categories. Roughly speaking, from such an original (associative unital) finite dimensional algebra  $\Lambda$ , which can be interpreted as  $K$ -category whose objects are the elements of a complete family of primitive idempotents, one constructs, by using covering techniques, a new small  $K$ -category or, equivalently, an algebra with enough idempotents  $\tilde{\Lambda}$  with a canonical functor  $\tilde{\Lambda} \rightarrow \Lambda$ . Although one generally loses the unital condition of the algebra in the process, the module category over  $\tilde{\Lambda}$  is much simpler than the one over  $\Lambda$  and the two module categories are related via well-behaved functors. The process has been fundamental to tackle classification of representation-finite algebras (see [18], [8], [3]).

Motivated by the power of coverings to deal with representation-theoretic problems and by our interest in studying the conditions of symmetry, periodicity and Calabi-Yau of finite dimensional mesh algebras (see [1]), we introduce and study in this paper the notion of pseudo-Frobenius graded algebra with enough idempotents. We show that such an algebra shares a lot of features of finite dimensional self-injective algebras. In particular, it can be characterized via the existence of a graded Nakayama form and it affords a Nakayama automorphism which is fundamental for its study. What is even more important, we show that, under fairly general hypotheses, the pseudo-Frobenius condition of a graded algebra with enough idempotents is preserved and reflected via covering functors and the associated graded Nakayama form and Nakayama automorphism move rather well from a given algebra to its covering and viceversa. This latter fact has been the key point for the results obtained in [1].

In this paper the term ‘algebra’ will mean always an associative algebra over a ground field  $K$  and all gradings on such algebra are  $H$ -gradings, where  $H$  is an abelian group with additive notation. Both  $K$  and  $H$  will be fixed in the rest of the paper. This is enough for our purposes, although some proofs and results might be adapted to more general situations.

The organization of the paper goes as follows. In section 2 we give the definition and basic properties of a graded algebra with enough idempotents, showing that it can be interpreted as a small graded  $K$ -category and viceversa. We characterize the graded Jacobson radical of such an algebra and define the concept of weakly basic graded algebra with enough idempotents, which is the analogue of a basic algebra in the ungraded unital setting. In section 3, as way of introduction of the concepts, we prove Theorem 3.1 which characterizes, in the weakly basic case, the pseudo-Frobenius graded algebras and shows that Quasi-Frobenius graded algebras with enough idempotents are always pseudo-Frobenius, the converse being true in the locally Noetherian case (corollary 3.3). The new concept is given via a graded Nakayama form, which comes with an associated Nakayama permutation of a weakly basic set of idempotents in the algebra and with a degree map from this set to  $H$ . It is shown in Proposition 3.2 that this permutation depends only on the algebra and not on the Nakayama form itself, while the analogous fact is not true for the degree map. In Proposition 3.7 we give an explicit way of constructing a graded Nakayama form of a split pseudo-Frobenius graded algebra. We end the section by showing that, when  $A$  is a graded pseudo-Frobenius algebra given by a  $\mathbb{Z}$ -graded quiver with homogenous relations in the standard way, one can choose a graded Nakayama form for  $A$  with constant degree map. In the final section 4, we adapt to the graded situation the theory of coverings, using the recent approach to the topic of [4] and [2], which is more general than the original one. The main result of the section (Theorem 4.7) states that, in the weakly basic locally bounded case and when the involved group of graded automorphisms acts freely on objects, the associated covering functor preserves and reflects the pseudo-Frobenius condition. As a consequence (see Corollary 4.8), one gets that the universal cover of a split basic finite dimensional self-injective algebra  $\Lambda$  is a (trivially graded) pseudo-Frobenius algebra and that any finite dimensional Galois covering of  $\Lambda$  is also a self-injective algebra.

## 2 Graded algebras with enough idempotents

### 2.1 Definition and basic properties

Recall that an associative algebra  $A$  is said to be an **algebra with enough idempotents**, when there is a family  $(e_i)_{i \in I}$  of nonzero orthogonal idempotents such that  $\oplus_{i \in I} e_i A = A = \oplus_{i \in I} A e_i$ . Any such family  $(e_i)_{i \in I}$  will be called a **distinguished family**. From now on in this paper  $A$  is an algebra with enough idempotents on which we fix a distinguished family of orthogonal idempotents.

All considered (left or right)  $A$ -modules are supposed to be unital. For a left (resp. right)

$A$ -module  $M$ , that means that  $AM = M$  (resp.  $MA = M$ ) or, equivalently, that we have a decomposition  $M = \oplus_{i \in I} e_i M$  (resp.  $M = \oplus_{i \in I} M e_i$ ) as  $K$ -vector spaces. We denote by  $A - \text{Mod}$  and  $\text{Mod} - A$  the categories of left and right  $A$ -modules, respectively.

The enveloping algebra of  $A$  is the algebra  $A^e = A \otimes A^{op}$ . Here and in the rest of the paper, unadorned tensor products are considered over  $K$ . The algebra  $A^e$  is also an algebra with enough idempotents and, when  $a, b \in A$ , we will denote by  $a \otimes b^o$  the corresponding element of  $A^e$ . The distinguished family of orthogonal idempotents with which we will work is the family  $(e_i \otimes e_j^o)_{(i,j) \in I \times I}$ . A left  $A^e$ -module  $M$  will be identified with an  $A$ -bimodule by putting  $axb = (a \otimes b^o)x$ , for all  $x \in M$  and  $a, b \in A$ . Similarly, a right  $A^e$ -module is identified with an  $A$ -bimodule by putting  $axb = x(b \otimes a^o)$ , for all  $x \in M$  and  $a, b \in A$ . In this way, we identify the three categories  $A^e - \text{Mod}$ ,  $\text{Mod} - A^e$  and  $A - \text{Mod} - A$ , where the last one is the category of unitary  $A$ -bimodules, which we will simply name 'bimodules'.

Let  $H$  be an abelian group with additive notation. An  $H$ -graded algebra with enough idempotents will be an algebra with enough idempotents  $A$ , together with an  $H$ -grading  $A = \oplus_{h \in H} A_h$ , such that one can choose a distinguished family of orthogonal idempotents which are homogeneous of degree 0. Such a family  $(e_i)_{i \in I}$  will be fixed from now on. We will denote by  $A - Gr$  (resp.  $Gr - A$ ) the category  $(H)$ -graded (always unital) left (resp. right) modules, where the morphisms are the graded homomorphisms of degree 0. A **locally finite dimensional left (resp. right) graded  $A$ -module** is a graded module  $M = \oplus_{h \in H} M_h$  such that, for each  $i \in I$  and each  $h \in H$ , the vector space  $e_i M_h$  (resp.  $M_h e_i$ ) is finite dimensional. Note that the definition does not depend on the distinguished family  $(e_i)$ . We will denote by  $A - lfdgr$  and  $lfdgr - A$  the categories of left and right locally finite dimensional graded modules.

Given a graded left  $A$ -module  $M$ , we denote by  $D(M)$  the subspace of the vector space  $\text{Hom}_K(M, K)$  consisting of the linear forms  $f : M \rightarrow K$  such that  $f(e_i M_h) = 0$ , for all but finitely many  $(i, h) \in I \times H$ . The  $K$ -vector space  $D(M)$  has a canonical structure of graded right  $A$ -module given as follows. The multiplication  $D(M) \times A \rightarrow D(M)$  takes  $(f, a) \rightsquigarrow fa$ , where  $(fa)(x) = f(ax)$  for all  $x \in M$ . Note that then one has  $fe_i = 0$ , for all but finitely many  $i \in I$ , and  $f = \sum_{i \in I} f e_i$ . Therefore  $D(M)$  is unital. On the other hand, if we put  $D(M)_h := \{f \in D(M) : f(M_k) = 0, \text{ for all } k \in H \setminus \{-h\}\}$ , we get a decomposition  $D(M) = \oplus_{h \in H} D(M)_h$  which makes  $D(M)$  into a graded right  $A$ -modules. Note that  $D(M)_h e_i$  can be identified with  $\text{Hom}_K(e_i M_{-h}, K)$ , for all  $(i, h) \in I \times H$ . We will call  $D(M)$  the **dual graded module** of  $M$ .

Recall that if  $M$  is a graded  $A$ -module and  $k \in H$  is any element, then we get a graded module  $M[k]$  having the same underlying ungraded  $A$ -module as  $M$ , but where  $M[k]_h = M_{k+h}$  for each  $h \in H$ . If  $M$  and  $N$  are graded left  $A$ -modules, then  $\text{HOM}_A(M, N) := \oplus_{h \in H} \text{Hom}_{A-Gr}(M, N[h])$  has a structure of graded  $K$ -vector space, where the homogeneous component of degree  $h$  is  $\text{HOM}_A(M, N)_h := \text{Hom}_{A-Gr}(M, N[h])$ , i.e.,  $\text{HOM}_A(M, N)_h$  consists of the graded homomorphisms  $M \rightarrow N$  of degree  $h$ . The following is an analogue of classical results for modules over associative rings with unit, whose proof can be easily adapted. It is left to the reader.

**Proposition 2.1.** *The assignment  $M \rightsquigarrow D(M)$  extends to an exact contravariant  $K$ -linear functor  $D : A - Gr \rightarrow Gr - A$  (resp.  $D : Gr - A \rightarrow A - Gr$ ) satisfying the following properties:*

1. *The maps  $\sigma_M : M \rightarrow D^2(M) := (D \circ D)(M)$ , where  $\sigma_M(m)(f) = f(m)$  for all  $m \in M$  and  $f \in D(M)$ , are all injective and give a natural transformation  $\sigma : 1_{A-Gr} \rightarrow D^2 := D \circ D$  (resp.  $\sigma : 1_{Gr-A} \rightarrow D^2 := D \circ D$ )*
2. *If  $M$  is locally finite dimensional then  $\sigma_M$  is an isomorphism*
3. *The restrictions of  $D$  to the subcategories of locally finite dimensional graded  $A$ -modules define mutually inverse dualities  $D : A - lfdgr \xrightarrow{\cong^{op}} lfdgr - A : D$ .*
4. *If  $M$  and  $N$  are a left and a right graded  $A$ -module, respectively, then there is an isomorphism of graded  $K$ -vector spaces*

$$\eta_{M,N} : \text{HOM}_A(M, D(N)) \rightarrow D(N \otimes_A M),$$

*which is natural on both variables.*

When  $A = \bigoplus_{h \in H} A_h$  and  $B = \bigoplus_{h \in H} B_h$  are graded algebras with enough idempotents, the tensor algebra  $A \otimes B$  inherits a structure of graded  $H$ -algebra, where  $(A \otimes B)_h = \bigoplus_{s+t=h} A_s \otimes B_t$ . In particular, this applies to the enveloping algebra  $A^e$  and, as in the ungraded case, we will identify the categories  $A^e - Gr$  (resp.  $Gr - A^e$ ) and  $A - Gr - A$  of graded left (resp. right)  $A^e$ -modules and graded  $A$ -bimodules.

**Remark 2.2.** If  $M$  is a graded  $A$ -bimodule and we denote by  $D({}_A M)$ ,  $D(M_A)$  and  $D({}_A M_A)$ , respectively, the duals of  $M$  as a left module, right module or bimodule, then  $D({}_A M_A) = D({}_A M) \cap D(M_A)$  and, in general,  $D({}_A M)$  and  $D(M_A)$  need not be the same vector subspace of  $\text{Hom}_K(M, K)$ . They are equal if the following two properties hold:

1. For each  $(i, h) \in I \times H$ , there are only finitely many  $j \in I$  such that  $e_i M_h e_j \neq 0$
2. For each  $(i, h) \in I \times H$ , there are only finitely many  $j \in I$  such that  $e_j M_h e_i \neq 0$ .

**Remark 2.3.** When  $H = 0$ , we have  $A - Gr = A - \text{Mod}$  and  $D(M) = \{f : M \rightarrow K : f(e_i M) = 0, \text{ for almost all } i \in I\}$ .

**Definition 1.** Let  $A = \bigoplus_{h \in H} A_h$  be a graded algebra with enough idempotents. It will be called **locally finite dimensional** when the regular bimodule  ${}_A A_A$  is locally finite dimensional, i.e., when  $e_i A_h e_j$  is finite dimensional, for all  $(i, j, h) \in I \times I \times H$ . Such a graded algebra  $A$  will be called **graded locally bounded** when the following two conditions hold:

1. For each  $(i, h) \in I \times H$ , the set  $I^{(i, h)} = \{j \in I : e_i A_h e_j \neq 0\}$  is finite
2. For each  $(i, h) \in I \times H$ , the set  $I_{(i, h)} = \{j \in I : e_j A_h e_i \neq 0\}$  is finite.

**Remark 2.4.** For  $H = 0$ , the just defined concepts are the familiar ones of locally finite dimensional and locally bounded, introduced in the language of  $K$ -categories by Gabriel and collaborators (see, e.g., [3]).

## 2.2 Graded algebras with enough idempotents versus graded $K$ -categories

In this subsection we remind the reader that graded algebras with enough idempotents can be looked at as small graded  $K$ -categories, and viceversa.

A category  $\mathcal{C}$  is a  $K$ -category if  $\mathcal{C}(X, Y)$  is a  $K$ -vector space, for all objects  $X, Y$ , and the composition map  $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$  is  $K$ -bilinear, for all  $X, Y, Z \in \text{Ob}(\mathcal{C})$ . If now  $H$  is a fixed additive abelian group, then  $\mathcal{C}$  is an  $(H-)$  *graded  $K$ -category* if  $\mathcal{C}(X, Y) = \bigoplus_{h \in H} \mathcal{C}_h(X, Y)$  is a graded  $K$ -vector space, for all  $X, Y \in \text{Obj}(\mathcal{C})$ , and the composition map restricts to a  $(K$ -bilinear) map

$$\mathcal{C}_h(Y, Z) \times \mathcal{C}_k(X, Y) \rightarrow \mathcal{C}_{h+k}(X, Z)$$

for any  $h, k \in H$ . There is an obvious definition of *graded functor (of degree zero)* between graded  $K$ -categories whose formulation we leave to the reader.

The prototypical example of graded  $K$ -category is  $(K, H) - GR = K - GR$ . Its objects are the  $H$ -graded  $K$ -vector spaces and we define  $\text{Hom}_{K-GR}(V, W) := \text{HOM}_K(V, W) = \bigoplus_{h \in H} \text{Hom}_{K-GR}(V, W[h])$  (see the paragraph preceding Proposition 2.1).

If  $A = \bigoplus_{h \in H} A_h$  is a graded algebra with enough idempotents, on which we fix a distinguished family  $(e_i)_{i \in I}$  of orthogonal idempotents of degree zero, then we can look at it as a small graded  $K$ -category. Indeed we put  $\text{Ob}(A) = I$ ,  $A(i, j) = e_i A e_j$  and take as composition map  $e_j A e_k \times e_i A e_j \rightarrow e_i A e_k$  the antimultiplication:  $b \circ a := ab$ .

Conversely, if  $\mathcal{C}$  is a small graded  $K$ -category then  $R = \bigoplus_{X, Y \in \text{Ob}(\mathcal{C})} \mathcal{C}(X, Y)$  is a graded  $K$ -algebra with enough idempotents, where the family of identity maps  $(1_X)_{X \in \text{Ob}(\mathcal{C})}$  is a distinguished family of homogeneous elements of degree zero. We will call  $R$  the *functor algebra associated to  $\mathcal{C}$* . Let  $GrFun(\mathcal{C}, K - GR)$  denote the category of graded  $K$ -linear covariant functors, with morphisms the  $K$ -linear natural transformations. To each object  $F$  in this category, we canonically associate a graded left  $R$ -module  $\mathcal{M}(F)$  as follows. The underlying graded  $K$ -vector space is  $\mathcal{M}(F) = \bigoplus_{C \in \text{Ob}(\mathcal{C})} F(C)$ . If  $f \in {}_1 Y R 1_X = \mathcal{C}(X, Y)$  and  $z \in F(Z)$ , then we define  $f \cdot z = \delta_{XZ} F(f)(z)$ , where

$\delta_{XZ}$  is the Kronecker symbol. Note that if  $x \in F(X)$  then  $f \cdot x$  is an element of  $F(Y)$ , and if  $f$  and  $x$  are homogeneous elements, then  $f \cdot x$  is homogeneous of degree  $\deg(f) + \deg(x)$ .

Conversely, given a graded left  $R$ -module  $M$ , we can associate to it a graded functor  $F_M : \mathcal{C} \rightarrow K - GR$  as follows. We define  $F_M(X) = 1_X M$ , for each  $X \in \text{Ob}(\mathcal{C})$ , and if  $f \in \mathcal{C}(X, Y) = 1_Y R 1_X$  is any morphism, then  $F_M(f) : F_M(X) \rightarrow F_M(Y)$  maps  $x \rightsquigarrow fx$ .

Given an object  $X$  of the graded  $K$ -category  $\mathcal{C}$ , the associated *representable functor* is the functor  $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow K - GR$  which takes  $Y \rightsquigarrow \mathcal{C}(X, Y)$ , for each  $Y \in \text{Ob}(\mathcal{C})$ . With an easy adaptation of the proof in the ungraded case (see, e.g., [6][Proposition II.2]), we get:

**Proposition 2.5.** *Let  $\mathcal{C}$  be a small  $(H)$ -graded  $K$ -category and let  $R$  be its associated functor algebra. Then the assignments  $F \rightsquigarrow \mathcal{M}(F)$  and  $M \rightsquigarrow F_M$  extend to mutually quasi-inverse equivalences of categories  $\text{GrFun}(\mathcal{C}^{op}, K - GR) \xleftarrow{\cong} R - GR$ . These equivalences restrict to mutually quasi-inverse equivalences  $\text{GrFun}(\mathcal{C}^{op}, K - \text{lfdGR}) \xleftarrow{\cong} R - \text{lfdgr}$ , where  $K - \text{lfdGR}$  denotes the full graded subcategory of  $K - GR$  consisting of the locally finite dimensional graded  $K$ -vector spaces.*

*These equivalences identify the finitely generated projective  $R$ -modules with the direct summands of representable functors.*

Due to the contents of the paper, we will freely move from the language of graded algebras with enough idempotents to that of small graded  $K$ -categories and viceversa. In particular, given graded algebras with enough idempotents  $A$  and  $B$ , we will say that  $F : A \rightarrow B$  is a graded functor if it so when we interpret  $A$  and  $B$  as small graded  $K$ -categories.

## 2.3 Weakly basic graded algebras with enough idempotents

Before introducing the main concept of this subsection, we give the following useful result. Recall that, when  $B = \oplus_{h \in H} B_h$  is a graded unital algebra, the intersection of its maximal graded left ideals coincides with the intersection of its maximal graded right ideals (see [16][Lemma I.7.4]). We denote by  $J^{gr}(B)$  this ideal.

**Proposition 2.6.** *Let  $A = \oplus_{h \in H} A_h$  be a graded algebra with enough idempotents, let  $(e_i)_{i \in I}$  be a distinguished family of primitive idempotents and, for each finite subset  $F \subseteq I$ , let us put  $e_F = \sum_{i \in F} e_i$ . For a graded vector subspace  $J = \oplus_{i,j} e_i J e_j$  be of  $A$ , the following assertions are equivalent:*

1.  *$J$  is the intersection of all maximal graded left ideals of  $A$ .*
2.  *$J$  is the intersection of all maximal graded right ideals of  $A$ .*
3. *A homogeneous element  $x \in e_i A_h e_j$  is in  $J$  if, and only if, the element  $e_j - yx$  is invertible in  $e_j A_0 e_j$ , for each  $y \in e_j A_{-h} e_i$ .*
4. *A homogeneous element  $x \in e_i A_h e_j$  is in  $J$  if, and only if, the element  $e_i - xy$  is invertible in  $e_i A_0 e_i$ , for each  $y \in e_j A_{-h} e_i$ .*
5.  *$J = \bigcup_{F \subseteq I, F \text{ finite}} J^{gr}(e_F A e_F)$ .*

*Proof.* We denote by  $J(k)$  the graded vector subspace of  $A$  given by assertion  $k$ , for  $k = 1, 2, 3, 4, 5$ . Note that  $J(k)$  is a graded left (resp. right) ideal of  $A$  for  $k = 1, 3$  (resp.  $k = 2, 4$ ) and it is a two-sided ideal for  $k = 5$  by the previous comments. We shall prove that  $J(1) = J(3) = J(5)$  and, by left-right symmetry, we will get also that  $J(2) = J(4) = J(5)$ .

*First step:*  $J(1) = J(3)$ . Put  $J(1) = J$  and  $J(3) = J'$  for simplicity. Note that if  $\mathbf{m}_j$  is a maximal ideal of  $e_j A_0 e_j$ , then  $\mathbf{a} = \mathbf{A} \mathbf{m}_j \oplus (\oplus_{i \neq j} A e_i)$  is a graded left ideal of  $A$  which does not contain  $e_j$ . We choose a maximal graded left ideal  $\mathbf{m}$  of  $A$  such that  $\mathbf{a} \subseteq \mathbf{m}$ . We then get inclusions  $\mathbf{m}_j \subseteq \mathbf{a} \cap e_j A_0 e_j \subseteq \mathbf{m} \cap e_j A_0 e_j \subsetneq e_j A_0 e_j$ , where the last inclusion is strict since  $e_j \notin \mathbf{m}$ . The maximality of  $\mathbf{m}_j$  then gives that  $\mathbf{m}_j = \mathbf{m} \cap e_j A_0 e_j$ . We then get that  $e_j J_0 e_j = J \cap e_j A_0 e_j$  is contained in the Jacobson radical  $J(e_j A_0 e_j)$  of  $e_j A_0 e_j$ . Suppose that  $x \in J \cap e_i A_h e_j$  and  $y \in e_j A_{-h} e_i$ . Then  $yx \in e_j J_0 e_j \subseteq J(e_j A_0 e_j)$ . This implies that  $e_j - yx$  is invertible in  $e_j A_0 e_j$ , and so  $J \subseteq J'$ .

Conversely, take  $x \in e_i J'_h e_j = J' \cap e_i A_h e_j$  and suppose that  $x \notin J$ . We then have a maximal graded left ideal  $\mathbf{m}$  of  $A$  such that  $x \notin \mathbf{m}$ , so that  $\mathbf{m} + Ax = A$ . In particular, we get  $e_j = z + yx$ , for some  $z \in \mathbf{m} \cap e_j A_0 e_j$  and some  $y \in e_j A_{-h} e_i$ . It follows that  $z$  is invertible in  $e_j A_0 e_j$ , which in turn implies that  $e_j \in \mathbf{m}$ . But then  $x = xe_j \in \mathbf{m}$ , which is a contradiction.

*Second step:*  $J(1)=J(5)$ . Put  $J = J(1)$  and  $J'' = J(5)$ . Let  $\mathbf{m}_F$  be a maximal graded left ideal of  $e_F A e_F$ . Then  $\mathbf{a} = (\oplus_{i \in I \setminus F} A e_i) \oplus A \mathbf{m}_F$  is a graded left ideal of  $A$  which does not contain  $e_F$ . Arguing as in the first step, we get a maximal graded left ideal  $\mathbf{m}$  of  $A$  such that  $\mathbf{m} \cap e_F A e_F = \mathbf{m}_F$ . We then get  $J \cap e_F A e_F \subseteq J^{gr}(e_F A e_F)$ , and so  $J = \bigcup_{F \subseteq I, F \text{ finite}} J \cap e_F A e_F \subseteq \bigcup_{F \subseteq I, F \text{ finite}} J^{gr}(e_F A e_F) = J''$ .

Note that if  $F \subset F'$  are finite subsets of  $I$ , then, by the properties of graded unital algebras and the fact that  $e_F A e_F = e_F(e_{F'} A e_{F'})e_F$ , we have that  $J^{gr}(e_F A e_F) = e_F J^{gr}(e_{F'} A e_{F'})e_F \subseteq J^{gr}(e_{F'} A e_{F'})$ . Let now  $x \in e_i J''_h e_j$  be a homogeneous element in  $J''$ . We can then choose a finite subset  $F \subseteq I$  such that  $i, j \in F$  and  $x \in J^{gr}(e_F A e_F)$ . By replacing  $A$  and  $J$  by  $e_F A e_F$  and  $J^{gr}(e_F A e_F)$ , respectively, the first step gives that, for each  $y \in e_j A_{-h} e_i$ , the element  $e_j - yx$  is invertible in  $e_j(e_F A e_F)e_j = e_j A e_j$ . That is, we have that  $x \in J(3) = J$ , so that the inclusion  $J'' \subseteq J$  also holds.  $\square$

The graded two-sided ideal  $J$  given by last proposition will be called the *graded Jacobson radical* of  $A$  and will be denoted by  $J^{gr}(A)$ .

**Definition 2.** A locally finite dimensional graded algebra with enough idempotents  $A = \oplus_{h \in H} A_h$  will be called *weakly basic* when it has a distinguished family  $(e_i)_{i \in I}$  of orthogonal homogeneous idempotents of degree 0 such that:

1.  $e_i A_0 e_i$  is a local algebra, for each  $i \in I$
2.  $e_i A e_j$  is contained in the graded Jacobson radical  $J^{gr}(A)$ , for all  $i, j \in I$ ,  $i \neq j$ .

It will be called *basic* when, in addition,  $e_i A_h e_i \subseteq J^{gr}(A)$ , for all  $i \in I$  and  $h \in H \setminus \{0\}$ .

We will use also the term '(weakly) basic' to denote any distinguished family  $(e_i)_{i \in I}$  of orthogonal idempotents satisfying the above conditions.

A weakly basic graded algebra with enough idempotents will be called *split* when  $e_i A_0 e_i / e_i J(A_0) e_i \cong K$ , for each  $i \in I$ .

**Proposition 2.7.** Let  $A = \oplus_{h \in H} A_h$  be a weakly basic graded algebra with enough idempotents and let  $(e_i)$  be a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:

1.  $J^{gr}(A)_0$  is the Jacobson radical of  $A_0$ .
2. Each indecomposable finitely generated projective graded left  $A$ -module is isomorphic to  $A e_i[h]$ , for some  $(i, h) \in I \times H$ . Moreover, if  $A e_i[h]$  and  $A e_j[k]$  are isomorphic in  $A - Gr$ , then  $i = j$  and, in case  $A$  is basic, also  $h = k$ .
3. Each finitely generated projective graded left  $A$ -module is a finite direct sum of graded modules of the form  $A e_i[h]$ , with  $(i, h) \in I \times H$ .
4. Each finitely generated graded left  $A$ -module has a projective cover in the category  $A - Gr$ .
5. Each finitely generated projective graded left  $A$ -module is the projective cover of a finite direct sum of graded-simple modules (=simple objects of the category  $A - Gr$ ).

Moreover, the left-right symmetric versions of these assertions also hold.

*Proof.* 1) Put  $J = J^{gr}(A)$ . We need to prove that  $e_i J_0 e_j = e_i J(A_0) e_j$ , for all  $i, j \in I$ . Let  $x \in e_i A_0 e_j$  be any element. By Proposition 2.6, we have that  $x \in e_i J_0 e_j$  if, and only if,  $e_j - yx$  is invertible in  $e_i A_0 e_i$ , for all  $y \in e_j A_0 e_i$ . On the other hand, if we consider  $A_0$  as a trivially graded algebra and we apply to it that same proposition, we get that this last condition is equivalent to saying that  $x \in e_i J^{gr}(A_0) e_j = e_i J(A_0) e_j$ .

The proof of the remaining assertions is entirely similar to the one for semiperfect (ungraded) associative algebras with unit (see, e.g., [11]) and here we only summarize the adaptation, leaving the details to the reader. For assertion 2), suppose that there is an isomorphism  $f : Ae_i[h] \xrightarrow{\cong} Ae_j[k]$  in  $A - Gr$ , with  $(i, h), (j, k) \in I \times H$ . The map  $\rho : e_i A_{k-h} e_j \rightarrow \text{Hom}_{A-Gr}(Ae_i[h], Ae_j[k])$ , given by  $\rho(x)(a) = ax$ , for all  $a \in Ae_i$ , is an isomorphism of  $K$ -vector spaces, so that  $f = \rho_x$ , for a unique  $x \in e_i A_{k-h} e_j$ . Similarly, there is a unique  $y \in e_j A_{h-k} e_i$  such that  $f^{-1} = \rho_y$ . We then get that  $yx = e_i$  and  $xy = e_j$ . If  $i \neq j$ , this is a contradiction since  $e_i Ae_j + e_j Ae_i \subseteq J^{gr}(A)$ . Therefore we necessarily have  $i = j$  and, in case  $A$  is basic, we also have  $h = k$  for otherwise we would have that  $yx = e_i \in J^{gr}(A)$ , which is absurd.

On the other hand, the map  $\rho : e_i A_0 e_i \rightarrow \text{End}_{A-Gr}(Ae_i[h])$  given above is an isomorphism of algebras. Therefore each  $Ae_i[h]$  has a local endomorphism algebra in  $A - Gr$ . Since each finitely generated graded left  $A$ -module is an epimorphic image of a finite direct sum of modules of the form  $Ae_i[h]$ , we conclude that the category  $A - grproj$  of finitely generated projective graded left  $A$ -module is a Krull-Schmidt one, with any indecomposable object isomorphic to some  $Ae_i[h]$ . This proves assertion 2 and 3.

As in the ungraded case, the fact that  $\text{End}_{A-Gr}(Ae_i[h])$  is a local algebra implies that  $J^{gr}(A)e_i[h]$  is the unique maximal graded submodule of  $Ae_i[h]$ . If  $S_i := Ae_i/J^{gr}(A)e_i$ , then  $S_i[h]$  is a graded-simple module, for each  $h \in H$ , and all graded-simple left modules are of this form, up to isomorphism. Since the projection  $Ae_i[h] \rightarrow S_i[h]$  is a projective cover in  $A - Gr$  we conclude that each graded-simple left  $A$ -module has a projective cover in  $A - Gr$ . From this argument we immediately get assertion 5, while assertion 4 follows as in the ungraded case.

Finally, the definition of weakly basic graded algebra is left-right symmetric, so that the last statement of the proposition also follows.  $\square$

### 3 Graded pseudo-Frobenius algebras

#### 3.1 Definition and characterizations

We look at  $K$  as an  $H$ -graded algebra such that  $K_h = 0$ , for  $h \neq 0$ . If  $V = \bigoplus_{h \in H} V_h$  is a graded  $K$ -vector space, then its dual  $D(V)$  gets identified with the graded  $K$ -vector space  $\bigoplus_{h \in H} \text{Hom}_K(V_h, K)$ , with  $D(V)_h = \text{Hom}_K(V_{-h}, K)$  for all  $h \in H$ .

**Definition 3.** Let  $V = \bigoplus_{h \in H} V_h$  and  $W = \bigoplus_{h \in H} W_h$  be graded  $K$ -vector spaces, where the homogeneous components are finite dimensional, and let  $d \in H$  be any element. A bilinear form  $(-, -) : V \times W \rightarrow K$  is said to be of degree  $d$  if  $(V_h, W_k) \neq 0$  implies that  $h + k = d$ . Such a form will be called nondegenerate when the induced maps  $W \rightarrow D(V)$  ( $w \rightsquigarrow (-, w)$ ) and  $V \rightarrow D(W)$  ( $v \rightsquigarrow (v, -)$ ) are bijective.

Note that, in the above situation, if  $(-, -) : V \times W \rightarrow K$  is a nondegenerate bilinear form of degree  $d$ , then the bijective map  $W \rightarrow D(V)$  (resp  $V \rightarrow D(W)$ ) given above gives an isomorphism of graded  $K$ -vector spaces  $W[d] \xrightarrow{\cong} D(V)$  (resp.  $V[d] \xrightarrow{\cong} D(W)$ ).

The following concept is fundamental for us.

**Definition 4.** Let  $A = \bigoplus_{h \in H} A_h$  be a weakly basic graded algebra with enough idempotents. A bilinear form  $(-, -) : A \times A \rightarrow K$  is said to be a graded Nakayama form when the following assertions hold:

1.  $(ab, c) = (a, bc)$ , for all  $a, b, c \in A$
2. For each  $i \in I$  there is a unique  $\nu(i) \in I$  such that  $(e_i A, Ae_{\nu(i)}) \neq 0$  and the assignment  $i \rightsquigarrow \nu(i)$  defines a bijection  $\nu : I \rightarrow I$ .
3. There is a map  $\mathbf{h} : I \rightarrow H$  such that the induced map  $(-, -) : e_i Ae_j \times e_j Ae_{\nu(i)} \rightarrow K$  is a nondegenerated graded bilinear form degree  $h_i = \mathbf{h}(i)$ , for all  $i, j \in I$ .

The bijection  $\nu$  is called the Nakayama permutation and  $\mathbf{h}$  will be called the degree map. When  $\mathbf{h}$  is a constant map and  $\mathbf{h}(i) = h$ , we will say that  $(-, -) : A \times A \rightarrow K$  is a graded Nakayama form of degree  $h$ .

**Definition 5.** A graded algebra with enough idempotents  $A = \bigoplus_{h \in H} A_h$  will be called left (resp. right) locally Noetherian when  $Ae_i$  (resp.  $e_i A$ ) satisfies ACC on graded submodules, for each  $i \in I$ . We will simply say that it is locally Noetherian when it is left and right locally Noetherian.

Recall that a graded module is *finitely cogenerated* when its graded socle is finitely generated and essential as a graded submodule. Recall also that a Quillen exact category  $\mathcal{E}$  (e.g. an abelian category) is said to be a *Frobenius category* when it has enough projectives and enough injectives and the projective and the injective objects are the same in  $\mathcal{E}$ .

**Theorem 3.1.** Let  $A = \bigoplus_{h \in H} A_h$  be a weakly basic graded algebra with enough idempotents. Consider the following assertions:

1.  $A - Gr$  and  $Gr - A$  are Frobenius categories
2.  $D({}_A A)$  and  $D(A_A)$  are projective graded  $A$ -modules
3. The projective finitely generated objects and the injective finitely cogenerated objects coincide in  $A - Gr$  (resp.  $Gr - A$ )
4. There exists a graded Nakayama form  $(-, -) : A \times A \longrightarrow K$ .

Then the following chain of implications holds:

$$1) \implies 2) \implies 3) \iff 4).$$

When  $A$  is graded locally bounded, also  $4) \implies 2)$  holds. Finally, if  $A$  is graded locally Noetherian, then the four assertions are equivalent.

*Proof.*  $1) \implies 2)$  By proposition 2.1, we have a natural isomorphism

$$HOM_A(?, D(A_A)) \cong D(A \otimes_A ?) : A - Gr \longrightarrow K - Gr,$$

and the second functor is exact. Then also the first is exact, which is equivalent to saying that  $D(A_A)$  is an injective object of  $A - Gr$  (see [16][Lemma I.2.4]). A symmetric argument proves that  $D({}_A A)$  is injective in  $Gr - A$ . Then both  $D(A_A)$  and  $D({}_A A)$  are projective in  $A - Gr$  and  $Gr - A$  since these are Frobenius categories.

$2) \implies 3)$  The duality  $D : A - lfdgr \xrightarrow{\cong^{op}} lfdgr - A : D$  exchanges projective and injective objects, and, also, simple objects on the left and on the right. Since  $A$  is locally finite dimensional all finitely generated left or right graded  $A$ -modules are locally finite dimensional. Moreover, our hypotheses guarantee that each finitely generated projective graded  $A$ -module  $P$  is the projective cover of a finite direct sum of simple graded modules. Then  $D(P)$  is the injective envelope in  $A - lfdgr$  of a finite direct sum of simple objects. We claim that each injective object  $E$  of  $A - lfdgr$  is an injective object of  $A - Gr$ . Indeed if  $U$  is a graded left ideal of  $A$ ,  $h \in H$  is any element and  $f : U[h] \longrightarrow E$  is morphism in  $A - Gr$ , then we want to prove that  $f$  extends to  $A[h]$ . By an appropriate use of Zorn lemma, we can assume without loss of generality that there is no graded submodule  $V$  of  $A[h]$  such that  $U[h] \subsetneq V$  and  $f$  is extendable to  $V$ . The task is then reduced to prove that  $U = A$ . Suppose this is not the case, so that there exist  $i \in I$  and a homogeneous element  $x \in Ae_i$  such that  $x \notin U$ . But then  $Ax + U/U$  is a locally finite dimensional graded  $A$ -module since so is  $Ax$ . It follows that  $\text{Ext}_{A-lfdgr}(\frac{U+Ax}{U}[h], E) = 0$ , which implies that  $f : U[h] \longrightarrow E$  can be extended to  $(U + Ax)[h]$ , thus giving a contradiction. Now the obvious graded version of Baer's criterion (see [16][Lema I.2.4]) holds and  $E$  is injective in  $A - Gr$ . In our situation, we conclude that  $D(P)$  is a finitely cogenerated injective object of  $A - Gr$ , for each finitely generated projective object  $P$  of  $Gr - A$ .

Conversely, if  $S$  is a simple graded right  $A$ -modules and  $p : P \longrightarrow D(S)$  is a projective cover, then  $D(p) : S \cong DD(S) \longrightarrow D(P)$  is an injective envelope. This proves that the injective envelope in  $A - Gr$  of any simple object, and hence any finitely cogenerated injective object of  $A - Gr$ , is locally finite dimensional.

Let now  $E$  be any locally finite dimensional graded left  $A$ -module. We then get that  $E$  is an injective finitely cogenerated object of  $A - Gr$  if, and only if,  $E \cong D(P)$  for some finitely generated projective graded right  $A$ -module  $P$ . This implies that  $E$  is isomorphic to a finite direct sum of graded modules of the form  $D(e_i A[-h_i]) \cong D(e_i A)[h_i]$ , where  $h_i \in H$ . We then assume,



without loss of generality, that  $E = D(e_i A)[h]$ , for some  $i \in I$  and  $h \in H$ . Since  $e_i A[-h]$  is a direct summand of  $A[-h]$  in  $Gr - A$ , assertion 2 implies that  $E$  is a projective object in  $A - Gr$ . Then  $E$  is isomorphic to a direct summand of a direct sum of graded modules of the form  $Ae_i[h_i]$ . From the fact that  $E$  has a finitely generated essential graded socle we easily derive that  $E$  is a direct summand of  $\bigoplus_{1 \leq k \leq r} Ae_{i_k}[h_{i_k}]$ , for some indices  $i_k \in I$ . Therefore each finitely cogenerated injective object of  $A - Gr$  is finitely generated projective. The analogous fact is true for graded right  $A$ -modules.

On the other hand, if  $P$  is a finitely generated projective graded left  $A$ -module, then  $D(P)$  is a finitely cogenerated injective object of  $Gr - A$  and, by the previous paragraph, we know that  $D(P)$  is finitely generated projective. We then get that  $P \cong DD(P)$  is finitely cogenerated in  $A - Gr$ .

3)  $\implies$  4) From assertion 3) we obtain its left-right symmetric statement by applying the duality  $D : A - lfdgr \xrightarrow{\cong^{op}} lfdgr - A : D$ , bearing in mind that an injective object in  $lfdgr - A$  is also injective in  $Gr - A$ . It follows that  $D(e_i A)$  is an indecomposable finitely generated projective left  $A$ -module, for each  $i \in I$ . We then get a unique index  $\nu(i) \in I$  and an  $h_i \in H$  such that  $D(e_i A) \cong Ae_{\nu(i)}[h_i]$ . We then have a map  $\nu : I \rightarrow I$ . By the same reason, given another  $j \in I$ , we have that  $D(Ae_j) \cong e_{\mu(j)} A[k_j]$ , for a unique  $\mu(j) \in I$  and  $k_j \in H$ . We then get

$$e_i A \cong DD(e_i A) \cong D(Ae_{\nu(i)}[h_i]) \cong D(Ae_{\nu(i)})[-h_i] \cong e_{\mu\nu(i)} A[k_{\nu(i)} - h_i],$$

and, by proposition 2.7, we conclude that  $\mu\nu(i) = i$ , for all  $i \in I$ . This and its symmetric argument prove that the maps  $\mu$  and  $\nu$  are mutually inverse.

We fix an isomorphism of graded left  $A$ -modules  $f_i : Ae_{\nu(i)}[h_i] \xrightarrow{\cong} D(e_i A)$ , for each  $i \in I$ . Then we get a bilinear map

$$e_i A \times Ae_{\nu(i)} \xrightarrow{1 \times f_i} e_i A \times D(e_i A) \xrightarrow{\text{can}} K.$$

Note that we have  $(a, cb) = f_i(cb)(a) = [cf_i(b)](a) = f_i(b)(ac) = (ac, b)$ , for all  $(a, b) \in e_i A \times Ae_{\nu(i)}$  and all  $c \in A$ . This bilinear form is clearly nondegenerate because  $e_i A$  is locally finite dimensional and, due to the duality  $D$ , the canonical bilinear form  $e_i A \times D(e_i A) \rightarrow K$  is nondegenerate, and actually graded of degree 0 since  $D(e_i A)_k = D(e_i A_{-k}) = \text{Hom}_K(e_i A_{-k}, K)$ , for each  $k \in H$ . On the other hand, if  $s, t \in H$  and  $a \in e_i A_s$  and  $b \in Ae_{\nu(i)}_t$  are homogeneous elements, then the degree of  $b$  in  $Ae_{\nu(i)}[h_i]$  is  $t - h_i$ . We get that  $(a, b) \neq 0$  if, and only if,  $s + (t - h_i) = 0$ . This shows that the given bilinear form is graded of degree  $h_i$ .

We then define an obvious bilinear form  $(-, -) : A \times A \rightarrow K$  such that  $(e_i A, Ae_j) = 0$ , whenever  $j \neq \nu(i)$ , and whose restriction to  $e_i A \times Ae_{\nu(i)}$  is the graded bilinear form of degree  $h_i$  given above, for each  $i \in I$ . Since  $(a, b) = \sum_{i, j \in I} (e_i a, be_j) = \sum_{i \in I} (e_i a, be_{\nu(i)})$ , we get that  $(ac, b) = (a, cb)$ , for all  $a, b, c \in A$ , and, hence, that  $(-, -) : A \times A \rightarrow K$  is a graded Nakayama form.

4)  $\implies$  3) Let  $(-, -) : A \times A \rightarrow K$  be a graded Nakayama form and let  $\nu : I \rightarrow I$  and  $\mathbf{h} : I \rightarrow H$  be the maps given in definition 4. We put  $\mathbf{h}(i) = h_i$ , for each  $i \in I$ . Since the restriction of  $(-, -) : e_i A \times Ae_{\nu(i)} \rightarrow K$  is a nondegenerate graded bilinear form of degree  $h_i$ , we get induced isomorphisms of graded  $K$ -vector spaces  $f_i : Ae_{\nu(i)}[h_i] \rightarrow D(e_i A)$  and  $g_i : e_{\nu^{-1}(i)} A[h_i] \rightarrow D(Ae_i)$ , where  $f_i(b) = (-, b) : x \rightsquigarrow (x, b)$  and  $g_i(a) = (a, -) : y \rightsquigarrow (a, y)$ . The fact that  $(ac, b) = (a, cb)$ , for all  $a, b, c \in A$  implies that  $f_i$  is a morphism in  $A - Gr$  and  $g_i$  is a morphism in  $Gr - A$ . Therefore the projective finitely generated objects and the injective finitely cogenerated objects coincide in  $A - lfdgr$  and  $lfdgr - A$ . By our comments about the graded Baer criterion, assertion 3 follows immediately.

3), 4)  $\implies$  2) We assume that  $A$  is graded locally bounded. The hypotheses imply that the injective finitely cogenerated objects of  $A - Gr$  and  $Gr - A$  are locally finite dimensional and they coincide with the finitely generated projective modules. But in this case  $A$  is locally finite dimensional both as a left and as a right graded  $A$ -module. Indeed, given  $i \in I$ , one has  $e_i A_h = \bigoplus_{j \in I} e_i A_h e_j$ . By the graded locally bounded condition of  $A$  almost all summands of this direct sum are zero. This gives that, for each  $(i, h) \in I \times H$ , the vector spaces  $e_i A_h$  is finite dimensional, whence, that  ${}_A A$  is in  $A - lfdgr$ . Similarly, we get that  $A_A \in A - lfdgr$ . It follows that  $D({}_A A)$  and  $D(A_A)$  are locally finite dimensional.

We claim that  $D(A_A)$  is isomorphic to  $\bigoplus_{i \in I} D(e_i A)$  which, together with assertion 3, will give that  $D(A_A)$  is a projective graded left  $A$ -module. This plus its symmetric argument will then finish

the proof. Indeed  $\oplus_{i \in I} D(e_i A)$  is identified with the set of  $f \in D(A_A)$  such that  $f(e_i A) = 0$ , for almost all  $i \in I$ . But we have  $D(A_A) = D({}_A A_A)$  (see Remark 2.2 and Definition 1). This means that, for each  $g \in D(A_A)$ , we have that  $g(e_i A_h e_j) = 0$  for almost all  $(i, j, h) \in I \times I \times H$ . We then get that  $g \in \oplus_{i \in I} D(e_i A)$ , so that  $D(A_A) = \oplus_{i \in I} D(e_i A)$ .

3), 4)  $\implies$  1) We assume that  $A$  is graded locally Noetherian. Then  $A-Gr$  and  $Gr-A$  are locally Noetherian Grothendieck categories, i.e., the Noetherian objects form a set (up to isomorphism) and generate both categories. Then each injective object in  $A-Gr$  or  $Gr-A$  is a direct sum of indecomposable injective objects and each direct sum of injective objects is again injective (see [6][Proposition IV.6 and Theorem IV.2]). Since, by hypothesis,  $Ae_i$  and  $e_i A$  are injective objects in  $A-Gr$  and  $Gr-A$ , respectively, we deduce that each projective object in any of these categories is injective.

On the other hand,  $Ae_i$  (resp.  $e_i A$ ) is a Noetherian object of  $A-Gr$  (resp.  $Gr-A$ ), which implies by duality that  $D(Ae_i)$  (resp.  $D(e_i A)$ ) is an artinian object of  $lfdgr-A$  (resp.  $A-lfdgr$ ), and hence also of  $Gr-A$  (resp.  $A-Gr$ ). But we have  $D(Ae_i) \cong e_{\nu^{-1}(i)} A[h_{\nu^{-1}(i)}]$  (resp.  $D(e_i A) \cong Ae_{\nu(i)}[h_i]$ ), where  $\nu$  is the Nakayama permutation. By the bijectivity of  $\nu$ , we get that all  $Ae_j$  and  $e_j A$  are Artinian (and Noetherian) objects, whence they have finite length. Therefore  $A-Gr$  and  $Gr-A$  have a set of generators of finite length, which easily implies that the graded socle of each object in these categories is a graded essential submodule. In particular, each injective object in  $A-Gr$  (resp.  $Gr-A$ ) is the injective envelope of its graded socle. But if  $\{S_t : t \in T\}$  is a family of simple objects of  $A-Gr$  (resp.  $Gr-A$ ) and  $\iota_t : S_t \rightarrow E(S_t)$  is an injective envelope in  $A-Gr$  (resp.  $Gr-A$ ), then the induced map  $\iota := \oplus_{t \in T} : \oplus_{t \in T} S_t \rightarrow \oplus_{t \in T} E(S_t)$  is an injective envelope in  $A-Gr$  (resp.  $Gr-A$ ) since the direct sum of injectives is injective. Since each  $E(S_t)$  is finitely cogenerated, whence projective by hypothesis, it follows that each injective object in  $A-Gr$  (resp.  $Gr-A$ ) is projective.  $\square$

**Definition 6.** A weakly basic locally finite dimensional graded algebra satisfying any of the conditions 3 and 4 of the previous proposition will be said to be a graded pseudo-Frobenius algebra. When  $A$  satisfies condition 1, it will be called graded Quasi-Frobenius.

By definition, if  $(-, -) : A \times A \rightarrow K$  is a graded Nakayama form for  $A$ , then its associated Nakayama permutation  $\nu$  and degree map  $\mathbf{h}$  are uniquely determined by  $(-, -)$ . However, one can naturally ask if they just depend on  $A$ . The following result gives an answer:

**Proposition 3.2.** Let  $A$  be a graded pseudo-Frobenius algebra, with  $(e_i)_{i \in I}$  as a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:

1.  $Soc_{gr}(AA) = Soc_{gr}(A_A)$ . We will denote this ideal by  $Soc_{gr}(A)$ .
2. All graded Nakayama forms for  $A$  have the same Nakayama permutation. It assigns to each  $i \in I$  the unique  $\nu(i) \in I$  such that  $e_i Soc_{gr}(A) e_{\nu(i)} \neq 0$ .
3. All graded Nakayama forms for  $A$  do not have the same degree map.

*Proof.* 1), 2) Let  $(-, -) : A \times A \rightarrow K$  be a graded Nakayama form for  $A$ . We have seen in the proof of the implication 4)  $\implies$  3) in theorem 3.1 that then  $D(e_i A) \cong Ae_{\nu(i)}[h_i]$ . Due to Proposition 2.7, we get that  $\nu(i)$  is independent of  $(-, -)$ . Moreover, by duality, we get an isomorphism  $e_i A \cong D(Ae_{\nu(i)})[-h_i]$ , which induces an isomorphism between the graded socles. But the graded socle of  $D(Ae_j)$  is isomorphic to  $S_j := e_j A / e_j J^{gr}(A)$ , for each  $j \in I$ . We then get that  $Soc^{gr}(e_i A) e_j [-h] \cong \text{Hom}_{A-Gr}(S_j[h], e_i Soc^{gr}(A)) = 0$ , for all  $j \neq \nu(i)$  and  $h \in H$ . Therefore  $\nu(i)$  is the unique element of  $I$  such that  $Soc^{gr}(e_i A) e_{\nu(i)} \neq 0$ .

Put  $S = Soc_{gr}(e_i A)$  for simplicity. The two-sided graded ideal  $AS$  is, as a graded right module, a sum of copies of graded-simple right  $A$ -modules of the form  $\frac{e_{\nu(i)} A}{e_{\nu(i)} J} [h]$ , where  $J = J^{gr}(A)$  and  $h \in I$ . In particular, we have  $AS \subseteq Soc_{gr}(A_A) = \oplus_{j \in I} Soc_{gr}(e_j A)$ . If we had  $AS \subsetneq S$ , we would have that  $Soc_{gr}(e_j A) \cap AS \neq 0$ , for some  $j \neq i$ . But we know that  $Soc_{gr}(e_j A) \cong \frac{e_{\nu(j)} A}{e_{\nu(j)} J} [-h_j]$ . It would follow that we have an isomorphism  $\frac{e_{\nu(j)} A}{e_{\nu(j)} J} [-h_j] \cong \frac{e_{\nu(i)} A}{e_{\nu(i)} J} [h]$ , for some  $h \in H$ . Taking projective covers in  $Gr-A$ , we then get that  $e_{\nu(j)} A [-h_j] \cong e_{\nu(i)} A [h]$ , for some  $h \in H$ . By Proposition 2.7, we would get that  $\nu(j) = \nu(i)$ , which would give that  $j = i$  and, hence, a contradiction. Therefore we have  $AS = S$  and, hence,  $S$  is a two-sided ideal.

To end the proof of assertions 1 and 2, we use the fact that  $\text{Soc}_{gr}(A)A$  is essential as a graded left ideal of  $A$ , because each  $e_i A$  is finitely cogenerated injective (see Theorem 3.1 and Definition 6). We then have a homogeneous element  $0 \neq w \in S \cap \text{Soc}_{gr}(A)A$  such that  $Aw$  is a graded-simple left  $A$ -module. Bearing in mind that  $Se_{\nu(i)} = S$ , we get that  $Aw = \text{Soc}_{gr}(Ae_{\nu(i)})$ . Therefore we have that  $\text{Soc}_{gr}(Ae_{\nu(i)}) \subseteq \text{Soc}_{gr}(e_i A)$ . The bijective condition of  $\nu$  then gives that  $\text{Soc}_{gr}(A)A \subseteq \text{Soc}_{gr}(A)A$ . The opposite inclusion follows by symmetry.

3) In the example 3.4(4) below, the map  $[-, -] : A \times A \rightarrow K$  given by  $[f, g] = (f, xg)$  is a graded Nakayama form of degree  $m - 1$  for  $A$ .  $\square$

The following result complements theorem 3.1 and gives a handy criterion, in the locally Noetherian case, for  $A$  to be graded Quasi-Frobenius.

**Corollary 3.3.** *Let  $A = \bigoplus_{h \in H} A_h$  be a weakly basic graded algebra with  $(e_i)_{i \in I}$  as a weakly basic distinguished family of orthogonal idempotents. The following assertions are equivalent:*

1.  *$A$  is locally Noetherian and the following conditions hold:*
  - (a) *For each  $i \in I$ ,  $Ae_i$  and  $e_i A$  have a simple essential socle in  $A - Gr$  and  $Gr - A$ , respectively*
  - (b) *There are bijective maps  $\nu, \nu' : I \rightarrow I$  such that  $\text{Soc}_{gr}(e_i A) \cong \frac{e_{\nu(i)} A}{e_{\nu(i)} J_{gr}(A)}[h_i]$  and  $\text{Soc}_{gr}(Ae_i) \cong \frac{Ae_{\nu'(i)}}{J_{gr}(A)e_{\nu'(i)}}[h'_i]$ , for certain  $h_i, h'_i \in H$*
2.  *$A$  is a locally Noetherian pseudo-Frobenius graded algebra.*
3.  *$A$  is graded Quasi-Frobenius*

*Proof.* 2)  $\implies$  3) is given by theorem 3.1.

3)  $\implies$  2) By hypothesis, in  $A - Gr$  every countable direct sum of injective envelopes of simple objects is an injective object. Then one proves that each  $Ae_i$  is a Noetherian object of  $A - Gr$  as in the ungraded unital case (see the proof of [11][Theorem 6.5.1]). By symmetry, one also gets that each  $e_i A$  is a Noetherian object of  $Gr - A$ .

2)  $\implies$  1) By duality and the fact that the finitely generated projective and finitely cogenerated injective graded modules coincide, we know that  $\text{Soc}_{gr}(e_i A)$  is a simple object of  $lfdgr - A$  which is an essential graded submodule of  $e_i A$ . If  $\nu$  is the Nakayama permutation, then, by proposition 3.2, we have that  $\text{Soc}_{gr}(e_i A)e_{\nu(i)} \neq 0$ . This implies that  $\text{Hom}_{Gr-A}(e_{\nu(i)} A[h], \text{Soc}_{gr}(e_i A)) \neq 0$ , for some  $h \in H$ . It follows that  $\text{Soc}_{gr}(e_i A) \cong \frac{e_{\nu(i)} A}{e_{\nu(i)} J_{gr}(A)}[h]$ . A left-right symmetric argument proves that the map  $\nu' = \nu^{-1}$  satisfies the condition in the statement.

1)  $\implies$  2) By definition of weakly basic,  $A$  is locally finite dimensional, so that  $Ae_i$  and  $e_i A$  are locally finite dimensional modules, for all  $i \in I$ . It then follows by duality that  $D(e_i A)$  is an Artinian object of  $A - Gr$ , for all  $i \in I$ . By the same reason, we get that  $D(e_i A)$  has a unique simple essential quotient in  $A - Gr$ , meaning that  $D(e_i A)$  has a unique maximal superfluous subobject in this category. By a classical argument, it follows that  $D(e_i A)$  has a projective cover in  $A - Gr$ , which is an epimorphism of the form  $p : Ae_j[h] \twoheadrightarrow D(e_i A)$ . It follows from this that  $D(e_i A)$  is a Noetherian object, whence, an object of finite length in  $A - Gr$  since it is a quotient of a Noetherian object. With this and its symmetric argument we get that all finitely cogenerated injective objects in  $A - Gr$  and  $Gr - A$  have finite length, which implies by duality that also the finitely generated projective objects have finite length.

The fact that  $\text{Soc}_{gr}(e_i A)$  is graded-simple implies that the injective envelope of  $e_i A$  in  $A - Gr$  is of the form  $\iota : e_i A \hookrightarrow E \cong D(Ae_j)[h]$ , while the projective cover of  $E$  is of the form  $p : e_k A[h'] \twoheadrightarrow E$ . Then  $\iota$  factors through  $p$  yielding a monomorphism  $u : e_i A \hookrightarrow e_k A[h']$ . But then the graded socles of  $e_i A$  and  $e_k A[h']$  are isomorphic. By condition 1.b) and the weakly basic condition, this implies that  $i = k$ . By comparison of graded composition lengths, we get that  $u$  is an isomorphism, which in turn implies that both  $p$  and  $\iota$  are also isomorphisms. Therefore all the  $e_i A$ , and hence all finitely generated projective objects, are finitely cogenerated injective objects of  $A - Gr$ . The left-right symmetry of assertion 1 implies that the analogous fact is true in  $Gr - A$ . Then, applying duality, we get that the finitely generated projective objects and the finitely cogenerated injective objects coincide in  $A - Gr$  and  $Gr - A$ .  $\square$

### 3.2 Some examples

**Examples 3.4.** *The following are examples of graded pseudo-Frobenius algebras over a field  $K$ :*

1. *When  $H = 0$  and  $A = \Lambda$  is a finite-dimensional self-injective algebra, which is equivalent to saying that  $\Lambda$  is quasi-Frobenius.*
2. *The group algebra  $KH$ , with  $(KH)_h = Kh$  for each  $h \in H$ , is graded simple both as a left and as a right graded module over itself. It immediately follows that  $KH$  is graded Quasi-Frobenius algebra.*
3. *When  $\Lambda$  is any finite-dimensional split basic algebra and  $A = \hat{\Lambda}$  is its repetitive algebra, in the terminology of [10], then  $A$  is a (trivially graded) quasi-Frobenius algebra with enough idempotents (see op.cit.[Chapter II]).*
4. *Take the  $\mathbb{Z}$ -graded algebra  $A = K[x, x^{-1}, y, z]/(y^2, z^2)$ , where  $\deg(x) = \deg(y) = \deg(z) = 1$ . Given any integer  $m$ , we have a canonical basis  $\mathcal{B}_m = \{x^m, x^{m-1}y, x^{m-1}z, x^{m-2}yz\}$  of  $A_m$ . Consider the graded bilinear form  $A \times A \rightarrow K$  of degree  $m$  identified by the fact that if  $f \in A_n$  and  $g \in A_{m-n}$ , then  $(f, g)$  is the coefficient of  $x^{m-2}yz$  in the expression of  $fg$  as a  $K$ -linear combination of the elements of  $\mathcal{B}_m$ . Then  $(-, -)$  is a graded Nakayama form for  $A$ , so that  $A$  is graded pseudo-Frobenius.*

### 3.3 The Nakayama automorphism and construction of Nakayama forms

We remind the reader that if  $\sigma$  and  $\tau$  are graded automorphisms of  $A$  (of zero degree), then we can form a graded  $A$ – $A$ –bimodule  ${}_{\sigma}A_{\tau}$  as follows. We take  ${}_{\sigma}A_{\tau} = A$  as a graded  $K$ -vector space and we define  $a \cdot x \cdot b = \sigma(a)x\tau(b)$ , for all  $a, x, b \in A$ .

**Corollary 3.5.** *Let  $A = \oplus_{h \in H} A_h$  be a graded pseudo-Frobenius algebra, let  $(e_i)_{i \in I}$  be a weakly basic distinguished family of orthogonal idempotents and let  $D(A) = D({}_A A_A)$  the dual graded bimodule of  ${}_A A_A$ . The following assertions hold:*

1. *There is an automorphism of (ungraded) algebras  $\eta : A \rightarrow A$ , which permutes the idempotents  $e_i$  and maps homogeneous elements of  $\bigcup_{i,j \in I} e_i A e_j$  onto homogeneous elements, such that  ${}_1 A_{\eta}$  is isomorphic to  $D(A)$  as an ungraded  $A$ -bimodule.*
2. *If the map  $\mathbf{h} : I \rightarrow H$  associated to the Nakayama form  $(-, -) : A \times A \rightarrow K$  takes constant value  $h$ , then  $\eta$  can be chosen to be graded and such that  $D(A)$  is isomorphic to  ${}_1 A_{\eta}[h]$  as graded  $A$ -bimodules.*

*Proof.* By definition,  $D(A)$  consists of the linear forms  $f : A \rightarrow K$  such that  $f(e_i A_h e_j) = 0$ , for all but finitely many triples  $(i, j, h) \in I \times I \times H$ . On the other hand, the left  $A$ -module  $\oplus_{i \in I} D(e_i A)$  can be identified with the space of linear forms  $g : A = \oplus_{i \in I} e_i A \rightarrow K$  such that  $g(e_i A) = 0$ , for almost all  $i \in I$ , and fixed any  $i \in I$ , the set of pairs  $(j, h) \in I \times H$  such that  $g(e_i A_h e_j) \neq 0$  is finite. It follows that, when viewed as  $K$ -subspaces of  $\text{Hom}_K(A, K)$ , we have  $D(A) = \oplus_{i \in I} D(e_i A)$  and, by a left-right symmetric argument, we also have  $D(A) = \oplus_{i \in I} D(A e_i)$ .

1) Let us fix a graded Nakayama form  $(-, -) : A \times A \rightarrow K$  and associated maps  $\nu : I \rightarrow I$  and  $\mathbf{h} : I \rightarrow H$ . The assignment  $b \rightsquigarrow (-, b)$  gives an isomorphism of graded  $A$ -modules  $A e_{\nu(i)}[h_i] \xrightarrow{\cong} D(e_i A)$ , where  $h_i = \mathbf{h}(i)$ , for each  $i \in I$ . By taking the direct sum of all these maps, we get an isomorphism of ungraded left  $A$ -modules  ${}_A A \rightarrow \oplus_{i \in I} D(e_i A) = D(A)$ . Therefore the assignment  $b \rightsquigarrow (-, b)$  actually gives an isomorphism  ${}_A A \xrightarrow{\cong} {}_A D(A)$ . Symmetrically, the assignment  $a \rightsquigarrow (a, -)$  gives an isomorphism  $A_A \xrightarrow{\cong} D(A)_A$ . It then follows that, given  $a \in A$ , there is a unique  $\eta(a) \in A$  such that  $(a, -) = (-, \eta(a))$ . This gives a  $K$ -linear map  $\eta : A \rightarrow A$  which, by its own definition, is bijective. Moreover, given  $a, b, x \in A$ , we get

$$(x, \eta(ab)) = (ab, x) = (a, bx) = (bx, \eta(a)) = (b, x\eta(a)) = (x\eta(a), \eta(b)) = (x, \eta(a)\eta(b)),$$

which shows that  $\eta(ab) = \eta(a)\eta(b)$ , for all  $a, b \in A$ . Therefore  $\eta$  is an automorphism of  $A$  as an ungraded algebra. Moreover if  $0 \neq a \in e_i A e_j$ , then  $(a, -)$  vanishes on all  $e_{i'} A e_{j'}$  except in  $e_j A e_{\nu(i)}$ . Therefore  $(-, \eta(a))$  does the same. By definition of the Nakayama form, we necessarily have

$\eta(a) \in e_{\nu(i)}Ae_{\nu(j)}$ . We claim that if  $a \in e_iAe_j$  is an element of degree  $h$ , then  $\eta(a)$  is an element of degree  $h+h_j-h_i$ . Indeed, let  $h' \in H$  be such that  $\eta(a)_{h'} \neq 0$ . Then  $(-, \eta(a)_{h'}) : e_jAe_{\nu(i)} \rightarrow K$  is a nonzero linear form which vanishes on  $e_jA_k e_{\nu(i)}$ , for all  $k \neq h_j-h'$ . Let us pick up  $x \in e_jA_{h_j-h'}e_{\nu(i)}$  such that  $(x, \eta(a)_{h'}) \neq 0$ . Then we have that  $(x, \eta(a)) = (x, \eta(a)_{h'}) \neq 0$ , due to the fact that  $(-, -) : e_jAe_{\nu(i)} \times e_{\nu(i)}Ae_{\nu(j)} \rightarrow K$  is a graded bilinear form of degree  $h_j$ . We then get that  $0 \neq (x, \eta(a)) = (a, x)$ , which implies that  $h + (h_j - h') = h_i$ , and hence that  $h' = h + (h_j - h_i)$ . Then  $h'$  is uniquely determined by  $a$ , so that  $\eta(a)$  is homogeneous of degree  $h + h_j - h_i$  as desired.

Putting  $a = e_i$  in the previous paragraph, we get that  $\eta(e_i) \in e_{\nu(i)}Ae_{\nu(i)}$  has degree 0, and then  $\eta(e_i)$  is an idempotent element of the local algebra  $e_{\nu(i)}A_0e_{\nu(i)}$ . It follows that  $\eta(e_i) = e_{\nu(i)}$ , for each  $i \in I$ .

Finally, we consider the  $K$ -linear isomorphism  $f : A \rightarrow D(A)$  which maps  $b \rightsquigarrow (-, b) = (\eta^{-1}(b), -)$ . We readily see that  $f$  is a homomorphism of left  $A$ -modules. Moreover, we have equalities

$$(a, b\eta(b')) = (ab, \eta(b')) = (b', ab) = (b'a, b) = [f(b)b'](a),$$

which shows that  $f$  is a homomorphism of right  $A$ -modules  $A_\eta \rightarrow$

$D(A)$ . Then  $f$  is an isomorphism  ${}_1A_\eta \xrightarrow{\cong} D(A)$ .

2) The proof of assertion 1 shows that if  $\mathbf{h}(i) = h$ , for all  $i \in I$ , then  $\eta$  is a graded automorphism of degree 0. Moreover, the isomorphism  $f : {}_1A_\eta \xrightarrow{\cong} D(A)$  is the direct sum of the isomorphisms of graded left  $A$ -modules  $f_i : Ae_{\nu(i)}[h] \xrightarrow{\cong} D(e_iA)$  which map  $b \rightsquigarrow (-, b)$ . It then follows that  $f$  is an isomorphism of graded bimodules  ${}_1A_\eta[h] \xrightarrow{\cong} D(A)$ .  $\square$

To finish this subsection, we will see that if one knows that  $A$  is split graded pseudo-Frobenius, then all possible graded Nakayama forms for  $A$  come in similar way. Recall that if  $V = \bigoplus_{h \in H} V_h$  is a graded vector space, then its *support*, denoted  $\text{Supp}(V)$ , is the set of  $h \in H$  such that  $V_h \neq 0$ .

**Proposition 3.6.** *Let  $A$  be a split pseudo-Frobenius graded algebra and  $(e_i)_{i \in I}$  a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:*

1. *If  $h_i \in \text{Supp}(e_i \text{Soc}_{gr}(A))$ , then  $\dim(e_i \text{Soc}_{gr}(A))_{h_i} = 1$*
2. *For a bilinear form  $(-, -) : A \times A \rightarrow K$ , the following statements are equivalent:*
  - (a)  *$(-, -)$  is a graded Nakayama form for  $A$*
  - (b) *There exist an element  $\mathbf{h} = (h_i) \in \prod_{i \in I} \text{Supp}(e_i \text{Soc}_{gr}(A))$  and a basis  $\mathcal{B}_i$  of  $e_iA_{h_i}e_{\nu(i)}$ , for each  $i \in I$ , such that:*
    - i.  $\mathcal{B}_i$  contains a (unique) element  $w_i$  of  $e_i \text{Soc}_{gr}(A)_{h_i}$
    - ii. *If  $a, b \in \bigcup_{i,j} e_iAe_j$  are homogeneous elements, then  $(e_iA_h, A_ke_j) = 0$  unless  $j = \nu(i)$  and  $h+k = h_i$*
    - iii. *If  $(a, b) \in e_iA_h \times A_{h_i-h}e_{\nu(i)}$ , then  $(a, b)$  is the coefficient of  $w_i$  in the expression of  $ab$  as a linear combination of the elements of  $\mathcal{B}_i$ .*

*Proof.* 1) Let us fix  $h_i \in \text{Supp}(e_i \text{Soc}_{gr}(A))$  and suppose that  $\{x, y\}$  is a linearly independent subset of  $e_i \text{Soc}_{gr}(A)_{h_i}$ . We then have  $xA = yA$  since  $e_i \text{Soc}_{gr}(A)$  is graded-simple. We get from this that also  $xA_0 = yA_0$ . By proposition 2.7, we know that  $J(A_0) = J^{gr}(A)_0$  and the split hypothesis on  $A$  implies that  $A_0 = J(A_0) \oplus (\bigoplus_{j \in I} Ke_j)$ . It follows that  $Kx = x(\bigoplus_{j \in I} Ke_j) = xA_0 = yA_0 = y(\bigoplus_{j \in I} Ke_j) = Ky$ , which contradicts the linear independence of  $\{x, y\}$ .

2) b)  $\implies$  a) By Proposition 3.2, the Nakayama permutation is completely determined by  $A$ . The given element  $\mathbf{h}$  is then interpreted as a map  $I \rightarrow H$ , which will be our degree function. Moreover, we claim that the induced graded bilinear form  $e_iAe_j \times e_jAe_{\nu(i)} \rightarrow K$  is nondegenerate. Indeed, let  $0 \neq a \in e_iA_h e_j$  be a homogeneous element. Then  $aA \cap e_i \text{Soc}_{gr}(A) \neq 0$  since  $e_i \text{Soc}_{gr}(A) = \text{Soc}_{gr}(e_iA)$  is essential in  $e_iA$ . Taking  $b \in e_jA_{h_i-h}e_{\nu(i)}$  homogeneous such that  $ab \in e_i \text{Soc}_{gr}(A)$ , from conditions b.i and b.iii we derive that  $(a, b) \neq 0$ .

It only remains to check that  $(ab, c) = (a, bc)$ , for all  $a, b, c$ . This easily reduces to the case when  $a, b, c$  are homogeneous and there are indices  $i, j, k$  such that  $a = e_i a e_j$ ,  $b = e_j b e_k$  and  $c = e_k c e_{\nu(i)}$ . But in this case, we have  $(a, bc) = (ab, c) = 0$  when  $\deg(a) + \deg(b) + \deg(c) \neq h_i$ .

On the other hand, by condition b.iii), if  $\deg(a) + \deg(b) + \deg(c) = h_i$  then  $(ab, c)$  and  $(a, bc)$  are both the coefficient of  $w_i$  in the expression of  $abc$  as linear combination of the elements of  $\mathcal{B}_i$ . So the equality  $(ab, c) = (a, bc)$  holds, for all  $a, b, c \in A$ .

a)  $\implies$  b) We first take a basis  $\mathcal{B}^0$  of  $A_0$  such that  $\mathcal{B}^0 = \{e_i : i \in I\} \cup (\mathcal{B}^0 \cap J(A_0))$  and  $\mathcal{B}^0 \subseteq \bigcup_{i,j \in I} e_i A_0 e_j$ . The graded Nakayama form gives by restriction a nondegenerate bilinear map

$$(-, -) : e_i A_0 e_i \times e_i A_{h_i} e_{\nu(i)} \longrightarrow K.$$

We choose as  $\mathcal{B}_i$  the basis of  $e_i A_{h_i} e_{\nu(i)}$  which is right orthogonal to  $e_i \mathcal{B}^0 e_i$  with respect to this form. As usual, if  $b \in e_i \mathcal{B}^0 e_i$ , we denote by  $b^*$  the element of  $\mathcal{B}_i$  such that  $(c, b^*) = \delta_{bc}$ , where  $\delta_{bc}$  is the Kronecker symbol. We then claim that  $w_i := e_i^*$  is in  $e_i \text{Soc}_{gr}(A)$ . This will imply that  $h_i \in \text{Supp}(\text{Soc}_{gr}(A))$  and, due to assertion 1, we will get also that  $w_i$  is the only element of  $e_i \text{Soc}_{gr}(A)_{h_i}$  in  $\mathcal{B}_i$ . Indeed suppose that  $w_i \notin e_i \text{Soc}_{gr}(A)$ . We then have  $a \in J^{gr}(A)$  such that  $aw_i \neq 0$ . Without loss of generality, we assume that  $a$  is homogeneous and that  $a = e_j a e_i$ , for some  $j \in I$ . Then  $0 \neq aw_i \in e_j A e_{\nu(i)}$ , which implies the existence of a homogeneous element  $b \in e_i A e_j$  such that  $(b, aw_i) \neq 0$  since the induced graded bilinear form  $e_i A e_j \times e_j A e_{\nu(i)} \longrightarrow K$  is nondegenerate. But then we have  $(ba, w_i) \neq 0$  and  $\deg(ba) = 0$  since the induced graded bilinear form  $e_i A e_i \times e_i A e_{\nu(i)} \longrightarrow K$  is of degree  $h_i$ . But  $ba \in e_i J^{gr}(A)_0 e_i = e_i J(A_0) e_i$  and, by the choice of the basis  $\mathcal{B}^0$ , each element of  $e_i J(A_0) e_i$  is a linear combination of the elements in  $\mathcal{B}^0 \cap e_i J(A_0) e_i$ . By the choice of  $w_i$ , we have  $(c, w_i) = 0$ , for all  $c \in \mathcal{B}^0 \cap e_i J(A_0) e_i$ . It then follows that  $(ba, w_i) = 0$ , which is a contradiction.

It is now clear that conditions b.i and b.ii hold. In order to prove b.iii, take  $(a, b) \in e_i A_{h_i} \times A_{h_i-h} e_{\nu(i)}$ . We then have  $(a, b) = (e_i, ab)$ , where  $ab \in e_i A_{h_i} e_{\nu(i)}$ . Put  $ab = \sum_{c \in \mathcal{B}_i} \lambda_c c$ , where  $\lambda_c \in K$  for each  $c \in \mathcal{B}_i$ . We then get  $(a, b) = (e_i, \sum_c \lambda_c c) = \sum_c \lambda_c (e_i, c) = \lambda_{w_i}$ , i.e.,  $(a, b)$  is the coefficient of  $w_i$  in the expression  $ab = \sum_c \lambda_c c$ .  $\square$

**Definition 7.** Let  $A = \oplus_{h \in H} A_h$  be a split pseudo-Frobenius graded algebra, with  $(e_i)_{i \in I}$  as weakly basic distinguished family of idempotents and  $\nu : I \longrightarrow I$  as Nakayama permutation. Given a pair  $(\mathcal{B}, \mathbf{h})$  consisting of an element  $\mathbf{h} = (h_i)_{i \in I}$  of  $\prod_{i \in I} \text{Supp}(e_i \text{Soc}_{gr}(A))$  and a family  $\mathcal{B} = (\mathcal{B}_i)_{i \in I}$ , where  $\mathcal{B}_i$  is a basis of  $e_i A_{h_i} e_{\nu(i)}$  containing an element of  $e_i \text{Soc}_{gr}(A)$ , for each  $i \in I$ , we call graded Nakayama form associated to  $(\mathcal{B}, \mathbf{h})$  to the bilinear form  $(-, -) : A \times A \longrightarrow K$  determined by the conditions b.ii and b.iii of last proposition. When  $\mathbf{h}$  is constant, i.e. there is  $h \in H$  such that  $h_i = h$  for all  $i \in I$ , we will call  $(-, -)$  the graded Nakayama form of  $A$  of degree  $h$  associated to  $\mathcal{B}$ .

### 3.4 Graded algebras given by quivers and relations

Recall that a *quiver* or *oriented graph* is a quadruple  $Q = (Q_0, Q_1, i, t)$ , where  $Q_0$  and  $Q_1$  are sets, whose elements are called *vertices* and *arrows* respectively, and  $i, t : Q_1 \longrightarrow Q_0$  are maps. If  $a \in Q_1$  then  $i(a)$  and  $t(a)$  are called the *origin* (or *initial vertex*) and the *terminus* of  $a$ .

Given a quiver  $Q$ , a path in  $Q$  is a concatenation of arrows  $p = a_1 a_2 \dots a_r$  such that  $t(a_k) = i(a_{k+1})$ , for all  $k = 1, \dots, r$ . In such case, we put  $i(p) = i(a_1)$  and  $t(p) = t(a_r)$  and call them the origin and terminus of  $p$ . The number  $r$  is the *length* of  $p$  and we view the vertices of  $Q$  as paths of length 0. The *path algebra* of  $Q$ , denoted by  $KQ$ , is the  $K$ -vector space with basis the set of paths, where the multiplication extends by  $K$ -linearity the multiplication of paths. This multiplication is defined as  $pq = 0$ , when  $t(p) \neq i(q)$ , and  $pq$  is the obvious concatenation path, when  $t(p) = i(q)$ . The algebra  $KQ$  is an algebra with enough idempotents, where  $Q_0$  is a distinguished family of orthogonal idempotents. If  $i \in Q_0$  is a vertex, we will write it as  $e_i$  when we view it as an element of  $KQ$ .

**Definition 8.** Let  $H$  be an abelian group. An  $(H)$ -graded quiver is a pair  $(Q, \deg)$ , where  $Q$  is a quiver and  $\deg : Q_1 \longrightarrow H$  is a map, called the degree or weight function.  $(Q, \deg)$  will be called locally finite dimensional when, for each  $(i, j, h) \in Q_0 \times Q_0 \times H$ , the set of arrows  $a$  such that  $(i(a), t(a), \deg(a)) = (i, j, h)$  is finite.

We will simply say that  $Q$  is an  $H$ -graded quiver, without mention to the degree function which is implicitly understood. Each degree function on a quiver  $Q$  induces an  $H$ -grading on the algebra  $KQ$ , where the degree of a path of positive length is defined as the sum of the degrees of its arrows and  $\deg(e_i) = 0$ , for all  $i \in Q_0$ . In the following result, for each natural number  $n$ , we denote by

$KQ_{\geq n}$  the vector subspace of  $KQ$  generated by the paths of length  $\geq n$ . For each ideal  $I$  of an algebra, we put  $I^\omega = \bigcap_{n>0} I^n$ .

**Proposition 3.7.** *Let  $A = \bigoplus_{h \in H} A_h$  be a split basic locally finite dimensional graded algebra with enough idempotents and let  $J = J^{gr}(A)$  be its graded Jacobson radical. There is an  $H$ -graded locally finite dimensional quiver  $Q$  and a subset  $\rho \subset \bigcup_{i,j \in Q_0} e_i KQ_{\geq 2} e_j$ , consisting of homogeneous elements with respect to the induced  $H$ -grading on  $KQ$ , such that  $A/J^\omega$  is isomorphic to  $KQ/\rho$ . Moreover  $Q$  is unique, up to isomorphism of  $H$ -graded quivers.*

*Proof.* It is an adaptation of the corresponding proof, in more restrictive situations, of the ungraded case (see, e.g., [3][Section 2]). We give the general outline, leaving the details to the reader.

Let  $(e_i)_{i \in I}$  be the basic distinguished family of orthogonal idempotents. The graded quiver  $Q$  will have  $Q_0 = I$  as its sets of vertices. Whenever  $h \in \text{Supp}(\frac{e_i J e_j}{e_i J^2 e_j})$ , we will select a subset  $Q_1(i, j)_h$  of  $e_i J_h e_j$  whose image by the projection  $e_i J_h e_j \rightarrow \frac{e_i J_h e_j}{e_i (J^2)_h e_j}$  gives a basis of  $\frac{e_i J_h e_j}{e_i (J^2)_h e_j}$ . We will take as arrows of degree  $h$  from  $i$  to  $j$  the elements of  $Q_1(i, j)_h$ , and then  $Q_1 = \bigcup_{i,j \in Q_0; h \in H} Q_1(i, j)_h$ . The so-obtained graded quiver gives a grading on  $KQ$  and there is an obvious homomorphism of graded algebras  $f : KQ \rightarrow A$  which takes  $e_i \rightsquigarrow e_i$  and  $a \rightsquigarrow a$ , for all  $i \in Q_0$  and  $a \in Q_1$ .

We claim that the composition  $KQ \xrightarrow{f} A \xrightarrow{p} A/J^\omega$  is surjective or, equivalently, that  $\text{Im}(f) + J^\omega = A$ . Due to the split basic condition of  $A$ , it is easy to see that  $A = (\sum_{i \in I} K e_i) \oplus J$  and the task is then reduced to prove the inclusion  $J \subseteq \text{Im}(f) + J^\omega$ . Since  $e_i A_h e_j$  is finite-dimensional, for each triple  $(i, h, j) \in I \times H \times I$ , there is a smallest natural number  $m_{ij}(h)$  such that  $e_i (J^n)_h e_j = e_i (J^{n+1})_h e_j$ , for all  $n \geq m_{ij}(h)$ . We will prove, by induction on  $k \geq 0$ , that  $e_i (J^{m_{ij}(h)-k})_h e_j \subseteq \text{Im}(f) + J^\omega$ , for all  $(i, h, j)$ , and then the inclusion  $J \subseteq \text{Im}(f) + J^\omega$  will follow. The case  $k = 0$  is trivial, by the definition of  $m_{ij}(h)$ . So we assume that  $k > 0$  in the sequel. Fix any triple  $(i, h, j) \in I \times H \times I$  and put  $n := m_{ij}(h) - k$ . If  $x \in e_i (J^n)_h e_j$  then  $x$  is a sum of products of the form  $x_1 x_2 \cdots x_n$ , where  $x_r$  is a homogeneous element in  $e_{i'} J e_{j'}$ , for some pair  $(i', j') \in I \times I$ . So it is not restrictive to assume that  $x = x_1 x_2 \cdots x_n$  is a product as indicated. By definition of the arrows of  $Q$ , each  $x_r$  admits a decomposition  $x_r = y_r + z_r$ , where  $y_r$  is a linear combination of arrows (of the same degree) and  $z_r \in J^2$ . It follows that  $x = y + z$ , where  $y$  is a linear combination of paths of length  $n$  and  $z \in e_i J^{n+1} e_j$ . Then  $y \in \text{Im}(f)$  and, by the induction hypothesis, we know that  $z \in \text{Im}(f) + J^\omega$ .

Proving that  $\text{Ker}(p \circ f) \subseteq KQ_{\geq 2}$  goes as in the ungraded case, as so does the proof of the uniqueness of  $Q$ . Both are left to the reader.  $\square$

A weakly basic locally finite dimensional algebra  $A$  will be called *connected* when, for each pair  $(i, j) \in I \times I$  there is a sequence  $i = i_0, i_1, \dots, i_n = j$  of elements of  $I$  such that, for each  $k = 1, \dots, n$ , either  $e_{i_{k-1}} A e_{i_k} \neq 0$  or  $e_{i_k} A e_{i_{k-1}} \neq 0$ . If  $Q$  is a graded quiver, we say that  $Q$  is a *connected graded quiver* when it is connected as an ungraded quiver. That is, when, for each pair  $(i, j) \in Q_0 \times Q_0$ , there is a sequence  $i = i_0, i_1, \dots, i_n = j$  of vertices such that there is an arrow  $i_{k-1} \rightarrow i_k$  or an arrow  $i_k \rightarrow i_{k-1}$ , for each  $k = 1, \dots, n$ .

**Corollary 3.8.** *Let  $A = \bigoplus_{n \geq 0} A_n$  be a split basic locally finite dimensional positively  $\mathbb{Z}$ -graded such that  $A_0$  is semisimple. Then there exists a  $\mathbb{Z}$ -graded quiver  $Q$  with  $\deg(a) > 0$  for all  $a \in Q_1$ , uniquely determined up to isomorphism of  $\mathbb{Z}$ -graded quivers, such that  $A$  is isomorphic to  $KQ/I$ , for a homogeneous ideal  $I$  of  $KQ$  such that  $I \subseteq KQ_{\geq 2}$ . If, moreover,  $A$  is connected and the equality  $A_n = A_1 \cdot \dots \cdot A_1$  holds for all  $n > 0$ , then the following assertions are equivalent:*

1.  *$A$  is graded pseudo-Frobenius*
2. *There exists a graded Nakayama form  $(-, -) : A \times A \rightarrow K$  with constant degree function.*

*In particular, the Nakayama automorphism  $\eta$  is always graded in this latter case.*

*Proof.* The point here is that if  $x \in J^n$  is a homogeneous element, then  $\deg(x) \geq n$ , which implies that  $J^\omega$  does not contain nonzero homogeneous elements and, hence, that  $J^\omega = 0$ . Then the first part of the statement is a direct consequence of proposition 3.7. Moreover, one easily sees that the connectedness of  $A$  is equivalent in this case to the connectedness of the quiver  $Q$ .

As for the second part, we only need to prove that if  $(-, -) : A \times A \rightarrow K$  is a graded Nakayama form, then its associated degree function is constant. The argument is inspired by [12][Proposition

3.2]. We consider that  $A = KQ/I$ , where  $Q$  is connected. The facts that  $A_0$  is semisimple and  $A_n = A_1 \cdot \dots \cdot A_1$ , for all  $n > 0$ , translate into the fact that the degree function  $\deg : Q_1 \rightarrow \mathbb{Z}$  takes constant value 1, so that the induced grading on  $KQ$  is the one by path length.

Let now  $\eta : A \rightarrow A$  be the Nakayama automorphism associated to  $(-, -)$ . If  $a : i \rightarrow j$  is any arrow in  $Q$ , then from corollary 3.5 we get that  $\eta(a)$  is a homogeneous element in  $e_{\nu(i)} J e_{\nu(j)} = e_{\nu(i)} A e_{\nu(j)}$ . Since obviously  $\deg(\eta(a)) \neq 0$ , we get that  $\deg(\eta(a)) \geq \deg(a)$ , which implies that  $\deg(\eta(x)) \geq \deg(x)$ , for each homogeneous element  $x \in \bigcup_{i,j \in Q_0} e_i A e_j$ . Let again  $a : i \rightarrow j$  be an arrow and put  $x = \eta^{-1}(a)$ . We claim that  $x$  is homogeneous of degree 1. Indeed, we have  $x = x_1 + x_2 + \dots + x_n$ , with  $\deg(x_k) = k$ , so that  $a = \eta(x) = \eta(x_1) + \eta(x')$ , where  $x' = \sum_{2 \leq k \leq n} x_k$  and, hence,  $\eta(x')$  is a sum of homogeneous elements of degrees  $\geq 2$ . It follows that  $a = \eta(x_1)$  and  $\eta(x') = 0$ , which, by the bijective condition of  $\eta$ , gives that  $x' = 0$ . Therefore  $x = x_1$  as desired.

The last paragraph implies that, for each pair  $(i, j) \in Q_0 \times Q_0$  such that there is an arrow  $i \rightarrow j$  in  $Q$ , there is a vector subspace  $V_{ij}$  of  $e_{\nu^{-1}(i)} K Q_1 e_{\nu^{-1}(j)}$  such that  $\eta|_{V_{ij}} : V_{ij} \rightarrow e_i K Q_1 e_j$  is a bijection. Let now  $\tilde{Q}$  be the subquiver of  $Q$  with the same vertices and with arrows those  $a \in Q_1$  such that  $\deg(\eta(a)) = 1$ . Then  $V_{ij} \subseteq e_{\nu^{-1}(i)} K \tilde{Q} e_{\nu^{-1}(j)}$  and  $\tilde{A} = \frac{K\tilde{Q} + I}{I}$  is a subalgebra of  $A = KQ/I$  such that the image of the restriction map  $\eta|_{\tilde{A}} : \tilde{A} \rightarrow A$  contains the vertices and the arrows (when viewed as elements of  $A$  in the obvious way). Note that  $\eta|_{\tilde{A}}$  is a homomorphism of graded algebras, which immediately implies that it is surjective and, hence, bijective. But then necessarily  $\tilde{A} = A$  for  $\eta$  is an injective map. We will derive from this that  $\deg(\eta(a)) = \deg(a)$ , for each  $a \in Q_1$ . Indeed, if  $\deg(\eta(a)) > 1$ , then  $\eta(a) = \eta(x)$ , for some homogeneous element  $x \in \tilde{A}$  of degree  $\deg(x) = \deg(\eta(a))$ . By the injective condition of  $\eta$ , we would get that  $a = x$ , which is a contradiction.

If now  $h : Q_0 \rightarrow \mathbb{Z}$  is the degree function associated to the graded Nakayama form, the proof of corollary 3.5 gives that  $h_{i(a)} = h_{t(a)}$ , for each  $a \in Q_1$ . Due to the connectedness of  $Q$ , we conclude that  $h$  is a constant function.  $\square$

## 4 Coverings of graded categories and preservation of the pseudo-Frobenius condition

### 4.1 Covering theory of graded categories

In this part we will present the basics of covering theory of graded categories or, equivalently, of graded algebras with enough idempotents. It is an adaptation of the classical theory (see [18], [8], [3]), where we incorporate more recent ideas of [4] and [2], where some of the constraining hypotheses of the initial theory disappear.

Let  $A = \bigoplus_{h \in H} A_h$  and  $B = \bigoplus_{h \in H} B_h$  be two locally finite dimensional graded algebras with enough idempotents, with  $(e_i)_{i \in I}$  and  $(\epsilon_j)_{j \in J}$  as respective distinguished families of homogeneous orthogonal idempotents of degree 0. Suppose that  $F : A \rightarrow B$  is a graded functor and that it is surjective on objects, i.e., for each  $j \in J$  there exists  $i \in I$  such that  $F(e_i) = \epsilon_j$ . To this functor one canonically associates the *pullup or restriction of scalars functor*  $F^p : B - Gr \rightarrow A - Gr$ . If  $X$  is a graded left  $B$ -module, then we put  $e_i F^p(X) = \epsilon_{F(i)} X$ , for all  $i \in I$ , and if  $a \in \bigcup_{i,i' \in I} e_i A e_{i'}$  and  $x \in F^p(X)$ , then  $ax := F(a)x$ . It has a left adjoint  $F_\lambda : A - Gr \rightarrow B - Gr$ , called the *pushdown functor*, whose precise definition will be given below in the case that we will need in this work.

The procedure of taking a weak skeleton gives rise to a graded functor as above. Indeed, suppose that  $A$  is as above and consider the equivalence relation  $\sim$  in  $I$  such that  $i \sim i'$  if, and only if,  $Ae_i$  and  $Ae_{i'}$  are isomorphic graded  $A$ -modules. If  $I_0$  is a set of representatives under this relation, then we can consider the full graded subcategory of  $A$  having as objects the elements of  $I_0$ . This amounts to take the graded subalgebra  $B = \bigoplus_{i,i' \in I_0} e_i A e_{i'}$ , which will be called the *weak skeleton* of  $A$ . If we denote by  $[i]$  the unique element of  $I_0$  such that  $i \sim [i]$ , then there are elements  $\xi_i \in e_i A_0 e_{[i]}$  and  $\xi_i^{-1} \in e_{[i]} A_0 e_i$  such that  $\xi_i \xi_i^{-1} = e_i$  and  $\xi_i^{-1} \xi_i = e_{[i]}$ . We fix  $\xi_i$  and  $\xi_i^{-1}$  from now on. By convention, we assume that  $\xi_{[i]} = e_{[i]}$ , for each  $[i] \in I_0$ . Now we get a surjective on objects graded functor  $F : A \rightarrow B$  which takes  $i \rightsquigarrow [i]$  on objects and if  $a \in e_i A e_{i'}$ , then  $F(a) = \xi_i^{-1} a \xi_{i'}$ . If we take  $P = \bigoplus_{i \in I_0} e_i A$  then  $P$  is an  $H$ -graded  $B - A$ -bimodule and the pullup functor is naturally isomorphic to the 'unitarization' of the graded Hom functor,  $AHOM_B(P, -) : B - Gr \rightarrow A - Gr$



(see subsection 2.1). It is an equivalence of categories and the pushout functor  $F_\lambda$  gets identified with  $P \otimes_A - : A - Gr \longrightarrow B - Gr$ , which, up to isomorphism, takes  $M \rightsquigarrow \bigoplus_{i \in I_0} e_i M$ .

**Definition 9.** Let  $A$  and  $B$  be as above. A graded functor  $F : A \longrightarrow B$  will be called a **covering functor** when it is surjective on objects and, for each  $(i, j, h) \in I \times J \times H$ , the induced maps

$$\begin{aligned} \bigoplus_{i' \in F^{-1}(j)} e_i A_h e_{i'} &\longrightarrow e_{F(i)} B_h e_j \\ \bigoplus_{i' \in F^{-1}(j)} e_{i'} A_h e_i &\longrightarrow e_j B_h e_{F(i)} \end{aligned}$$

are bijective.

We shall now present the paradigmatic example of covering functor, which is actually the only one that we will need in our work. In the rest of this subsection,  $A = \bigoplus_{h \in H} A_h$  will be a locally finite dimensional graded algebra with a distinguished family  $(e_i)_{i \in I}$  of homogeneous orthogonal idempotents of degree 0, fixed from now on. We will assume that  $G$  is a group acting on  $A$  as a group of graded automorphisms (of degree 0) which permutes the  $e_i$ . That is, if  $\text{Aut}^{gr}(A)$  denotes the group of graded automorphisms of degree 0 which permute the  $e_i$ , then we have a group homomorphism  $\varphi : G \longrightarrow \text{Aut}^{gr}(A)$ . We will write  $a^g = \varphi(g)(a)$ , for each  $a \in A$  and  $g \in G$ . In such a case, the *skew group algebra*  $A \star G$  has as elements the formal  $A$ -linear combinations  $\sum_{g \in G} a_g \star g$ , with  $a_g \in A$  for all  $g \in G$ .

The multiplication extends by linearity the product  $(a \star g)(b \star g') = ab^g \star gg'$ , where  $a, b \in A$  and  $g, g' \in G$ . The new algebra inherits an  $H$ -grading from  $A$  by taking  $(A \star G)_h = A_h \star G = \{\sum_{g \in G} a_g \star g \in A \star G : a_g \in A_h, \text{ for all } g \in G\}$ . We have a canonical inclusion of  $H$ -graded algebras  $\iota : A \hookrightarrow A \star G$  which maps  $a \rightsquigarrow a \star 1$ , where 1 is the unit of  $G$ .

**Proposition 4.1.** In the situation above, let  $\Lambda$  be the weak skeleton of  $A \star G$  and  $F : A \star G \longrightarrow \Lambda$  the corresponding functor. The following assertions hold:

1.  $(A \star G)(e_i \star 1)$  and  $(A \star G)(e_j \star 1)$  (resp.  $(e_i \star 1)(A \star G)$  and  $(e_j \star 1)(A \star G)$ ) are isomorphic in  $A \star G - Gr$  (resp.  $Gr - A \star G$ ) if, and only if,  $i$  and  $j$  are in the same  $G$ -orbit.
2. The composition  $A \xrightarrow{\iota} A \star G \xrightarrow{F} \Lambda$  is a covering functor. The corresponding pushdown functor  $F_\lambda : A - Gr \longrightarrow \Lambda - Gr$  is exact and takes  $Ae_i \rightsquigarrow \Lambda e_{[i]}$ , for each  $i \in I$ .

*Proof.* All through the proof, whenever necessary, we identify  $a$  with  $a \star 1$ , for each  $a \in A$ .

1) We do the part of the assertion concerning graded left modules, leaving the one for right modules to the reader. We have that  $(A \star G)e_i \cong (A \star G)e_j$  if, and only if, there are  $x \in e_i(A \star G)_0 e_j = \bigoplus_{g \in G} e_i A_0 e_{g(j)} \star g$  and  $y \in e_j(A \star G)_0 e_i = \bigoplus_{g \in G} e_j A_0 e_{g(i)} \star g$  that  $xy = e_i$  and  $yx = e_j$ . This immediately implies that  $i$  and  $j$  are in the same  $G$ -orbit, i.e., that  $e_i^g = e_j$ , for some  $g \in G$ . Conversely, we have equalities  $e_i \star g = (e_i \star 1)(e_i \star g)(e_{g^{-1}(i)} \star 1)$  and  $e_{g^{-1}(i)} \star g^{-1} = (e_{g^{-1}(i)} \star 1)(e_{g^{-1}(i)} \star g^{-1})(e_i \star 1)$ , and also  $(e_i \star g)(e_{g^{-1}(i)} \star g^{-1}) = e_i \star 1$  and  $(e_{g^{-1}(i)} \star g^{-1})(e_i \star g) = e_{g^{-1}(i)} \star 1$ , which show that  $(A \star G)e_i \cong (A \star G)e_{g^{-1}(i)}$  for all  $g \in G$  and  $i \in I$ .

2) The pullup functor is the composition  $\Lambda - Gr \xrightarrow{F^\rho} A \star G - Gr \xrightarrow{\iota^\rho} A - Gr$ , so that the pushdown functor is  $F_\lambda \circ \iota_\lambda$ . We know that  $F_\lambda$  is an equivalence of categories. On the other hand  $\iota_\lambda$  is naturally isomorphic to  $A \star G \otimes_A - : A - Gr \longrightarrow A \star G - Gr$  since  $\iota^\rho$  is the usual restriction of scalars. The exactness of  $A \star G \otimes_A -$  implies that of  $F_\lambda \circ \iota_\lambda$  and the action of this functor on projective objects takes  $Ae_i \rightsquigarrow (A \star G) \otimes_A Ae_i \cong (A \star G)e_i \rightsquigarrow F_\lambda((A \star G)e_i)$ . But this latter graded  $\Lambda$ -module is isomorphic to  $\Lambda e_{[i]}$  by the explicit definition of the pushdown functor when taking a weak skeleton.

In order to check that  $F \circ \iota$  is a covering functor we look at the definition of the weak skeleton, taking into account assertion 1. We then take exactly one index  $i \in I$  in each  $G$ -orbit and in that way we get a subset  $I_0$  of  $I$ . Up to graded isomorphism, we have  $\Lambda = \bigoplus_{i, j \in I_0} e_i(A \star G)e_j$ . For the explicit definition of  $F$ , we put  $\xi_{g(i)} = e_{g(i)} \star g$  and  $\xi_{g(i)}^{-1} = e_i \star g^{-1}$ , for each  $i \in I_0$  and  $g \in G$ . If  $g, g' \in G$  and  $i, j \in I_0$ , then the map  $F : e_{g(i)}(A \star G)e_{g'(j)} \longrightarrow e_i \Lambda e_j = e_i(A \star G)e_j$  takes  $x \rightsquigarrow \xi_{g(i)}^{-1} x \xi_{g'(j)} = (e_i \star g^{-1})x(e_{g'(j)} \star g')$ . Then the composition

$$e_{g(i)} A e_{g'(j)} \xrightarrow{\iota} e_{g(i)}(A \star G)e_{g'(j)} \xrightarrow{F} e_i \Lambda e_j = e_i(A \star G)e_j$$

takes  $a \rightsquigarrow (e_i \star g^{-1})(a \star 1)(e_{g'(j)} \star g') = a^{g^{-1}} \star g^{-1} g'$ .

The proof that  $F \circ \iota$  is a covering functor gets then reduced to check that if  $i, j \in I_0$  and  $h \in H$  then the maps

$$\begin{aligned} \oplus_{g \in G} e_{g(i)} A_h e_j &\longrightarrow e_i \Lambda_h e_j = e_i (A \star G)_h e_j, & (a_g)_{g \in G} &\rightsquigarrow \sum_{g \in G} a_g^{g^{-1}} \star g^{-1} \\ \oplus_{g \in G} e_i A_h e_{g(j)} &\longrightarrow e_i \Lambda_h e_j = e_i (A \star G)_h e_j, & (b_g)_{g \in G} &\rightsquigarrow \sum_{g \in G} b_g \star g \end{aligned}$$

are both bijective. It is clear since  $\oplus_{g \in G} (e_{g(i)} A_h e_j)^{g^{-1}} \star g^{-1} = e_i (A \star G)_h e_j = \oplus_{g \in G} e_i A_h e_{g(j)} \star g$ .  $\square$

**Definition 10.** If  $A = \oplus_{h \in H} A_h$ ,  $G$  and  $\Lambda$  are as above, then the functor  $F \circ \iota : A \longrightarrow \Lambda$  will be called a  $G$ -covering of  $\Lambda$ .

If  $A$  and  $G$  are as in the setting, we say that  $G$  acts freely on objects when  $g(i) \neq i$ , for all  $i \in I$  and  $g \in G \setminus \{1\}$ . In such case we can form the orbit category  $A/G$ . The objects of this category are the  $G$ -orbits  $[i]$  of indices  $i \in I$  and the morphisms from  $[i]$  to  $[j]$  are formal sums  $\sum_{g \in G} [a_g]$ , where  $[a_g]$  is the  $G$ -orbit of an element  $a_g \in e_i A e_{g(j)}$ . This definition does not depend on  $i, j$ , but just on the orbits  $[i], [j]$ . The anticomposition of morphisms extends by  $K$ -linearity the following rule. If  $a, b \in \bigcup_{i, j \in I} e_i A e_j$  and  $[a], [b]$  denote the  $G$ -orbits of  $a$  and  $b$ , then  $[a] \cdot [b] = 0$ , in case  $[t(a)] \neq [i(b)]$ , and  $[a] \cdot [b] = [ab^g]$ , in case  $[t(a)] = [i(b)]$ , where  $g$  is the unique element of  $G$  such that  $g(i(b)) = t(a)$ . We have an obvious canonical projection  $\pi : A \longrightarrow A/G$  with takes  $a \rightsquigarrow [a]$ . The following is the classical interpretation of  $\Lambda$  and is implicit in [2].

**Corollary 4.2.** Let  $A, G$  and  $\Lambda$  be as in proposition 4.1 and suppose that  $G$  acts freely on objects. There is an equivalence of categories  $\Upsilon : \Lambda \xrightarrow{\cong} A/G$  such that  $\Upsilon \circ F \circ \iota : A \longrightarrow A/G$  is the canonical projection.

*Proof.* Let fix a set  $I_0$  of representatives of the elements of  $I$  under the equivalence relation  $\sim$  given by:  $i \sim j$  if, and only if,  $(A \star G)e_i \cong (A \star G)e_j$  are isomorphic as graded  $(A \star G)$ -modules. Then, by definition,  $\Lambda$  is the category having as objects the elements of  $I_0$  and  $e_i \Lambda e_j = e_i (A \star G) e_j = \oplus_{g \in G} [e_i A e_{g(j)} \star g]$  as space of morphisms from  $i$  to  $j$ . The functor  $\Upsilon : \Lambda \longrightarrow A/G$  is defined as  $\Upsilon(i) = [i]$ , for each  $i \in I_0$ , and by  $\Upsilon(a \star g) = [a]$ , when  $g \in G$  and  $a \in e_i A e_{g(j)}$ , with  $i, j \in I_0$ .

The functor is clearly dense. On the other hand, if  $\Upsilon(\sum_{g \in G} a_g \star g) = \Upsilon(\sum_{g \in G} b_g \star g)$ , with  $a_g, b_g \in e_i A e_{g(j)}$  for some  $i, j \in I_0$ , then we have an equality of formal finite sums of orbits  $\sum_{g \in G} [a_g] = \sum_{g \in G} [b_g]$ . This implies that  $[a_g] = [b_g]$ , for each  $g \in G$ , because if there is an element  $\sigma \in G$  such that  $\sigma(a_g)$  and  $b_h$  have the same origin and terminus, for some  $h \in G$ , then  $\sigma = id$  due to the free action on objects. But the equality  $[a_g] = [b_g]$  also implies that  $a_g = b_g$  since  $i(a_g) = i(b_g) = i$ . Therefore  $\Upsilon$  is a faithful functor. Finally, the orbit of any homogeneous morphism  $a$  in  $A$  contains an element with origin, say  $i$ , in  $I_0$ . Then, in order to prove that  $\Upsilon$  is full, we can assume that  $[a]$  is the orbit of an element  $a \in e_i A e_{g(j)}$ , for some  $i, j \in I_0$  and some  $g \in G$ . But then  $a \star g \in e_i (A \star G) e_j$ , and we clearly have  $\Upsilon(a \star g) = [a]$ .

The equality of functors  $\Upsilon \circ F \circ \iota = \pi$  is straightforward.  $\square$

## 4.2 Preservation of the pseudo-Frobenius condition

In this final subsection we will see that the pseudo-Frobenius condition is preserved and reflected by covering functors, under fairly general conditions. Recall that a graded algebra with enough idempotents is *graded semisimple* when as a (left or right) graded module over itself it is a direct sum of graded-simple modules. The following is a useful auxiliary result.

**Lemma 4.3.** Let  $A$  be a weakly basic  $(H)$ -graded locally bounded algebra, where  $(e_i)_{i \in I}$  is a weakly basic distinguished family of orthogonal idempotents, let  $G$  be a group of graded automorphisms of  $A$  which acts freely on objects and let  $A \star G$  the skew group algebra with its natural  $H$ -grading. The following assertions hold:

1. The graded Jacobson radical of  $A \star G$  is  $J^{gr}(A) \star G := \{\sum_{g \in G} a_g \star g \in A \star G : a_g \in J^{gr}(A), \text{ for all } g \in G\}$ .
2.  $\frac{A \star G}{J^{gr}(A \star G)}$  is isomorphic to the skew group algebra  $\frac{A}{J^{gr}(A)} \star G$ , which is graded semisimple.

3. If  $S$  is a graded-semisimple left (resp. right)  $A$ -module, then  $(A \star G) \otimes_A S$  (resp.  $S \otimes_A (A \star G)$ ) is a graded-semisimple left (resp. right)  $A \star G$ -module.

*Proof.* We will identify  $a$  with  $a \star 1$ , when we want to see the element  $a \in A$  as an element of  $A \star G$ .

1), 2) Let us put  $J = J^{gr}(A)$  and let  $a \in e_i J_h e_j$  be a homogeneous element. We claim that  $a \in J^{gr}(A \star G)$ , from which we will derive that  $J = J \star 1 \subset J^{gr}(A \star G)$ . It will follow that  $J \star G = (J \star 1)(A \star G) \subseteq J^{gr}(A \star G)$  since  $J^{gr}(A \star G)$  is a two-sided ideal of  $A \star G$ . We will prove that  $a(A \star G)_{-h} e_i$  consists of nilpotent elements in the algebra  $e_i(A \star G)_0 e_i$ , which will imply that, for each  $y \in e_j(A \star G)_{-h} e_i = \oplus_{g \in G} [e_j A_{-h} e_{g(i)} \star g]$ , the element  $e_i - ay = e_1 \star 1 - (a \star 1)y$  is invertible in  $e_i(A \star G)_0 e_i$ , thus settling our claim (see Proposition 2.6). Let us write  $y = y_1 \star g_1 + \dots + y_r \star g_r$ , where  $y_k \in e_j A_{-h} e_{g_k(i)}$  for  $k = 1, \dots, r$ , so that  $ay = \sum_{1 \leq k \leq r} ay_k \star g_k$ . Note that  $ay_k \in e_i A_0 e_{g_k(i)}$ , for each  $k = 1, \dots, r$ . Due to the locally bounded condition of  $A$  and the free action of  $G$  on objects, the set  $\{g \in G : e_i A_0 e_{g(i)} \neq 0\}$  is finite. Fix now an integer  $m > 0$  and consider, for each  $n > 0$ , the sequence of products  $\{(ay_{i_1})(ay_{i_2})^{g_{i_1}}, \dots, (ay_{i_1})(ay_{i_2})^{g_{i_1}} \cdot \dots \cdot (ay_{i_n})^{g_{i_1} g_{i_2} \dots g_{i_{n-1}}}\}$ . Bearing in mind that  $(ay_{i_1})(ay_{i_2})^{g_{i_1}} \cdot \dots \cdot (ay_{i_k})^{g_{i_1} g_{i_2} \dots g_{i_{k-1}}} \in e_i A_0 e_{g_{i_1} g_{i_2} \dots g_{i_{k-1}} g_{i_k}(i)}$ , for each  $k > 1$ , and that  $G$  acts freely on objects, we can always choose  $n$  large enough so that, for any choice of  $i_1, \dots, i_n \in \{1, 2, \dots, r\}$ , there is an index  $j$  in the  $G$ -orbit of  $i$ , such that at least  $m+1$  products in the last sequence are in  $e_i A_0 e_j$ . This implies that  $(ay_{i_1})(ay_{i_2})^{g_{i_1}} \cdot \dots \cdot (ay_{i_n})^{g_{i_1} g_{i_2} \dots g_{i_{n-1}}}$  is in  $e_i A_0 e_j \cdot (e_j J_0 e_j)^m$ . But, by proposition 2.7, we know that  $e_j J_0 e_j$  is the Jacobson radical of the finite dimensional algebra  $e_j A_0 e_j$ . Choosing  $m$  such that  $(e_j J_0 e_j)^m = 0$ , we conclude that it is always possible to choose a large enough  $n > 0$  such that  $(ay_{i_1})(ay_{i_2})^{g_{i_1}} \cdot \dots \cdot (ay_{i_n})^{g_{i_1} g_{i_2} \dots g_{i_{n-1}}} = 0$ , for any choice of  $i_1, \dots, i_n \in \{1, 2, \dots, r\}$ . It then follows that  $(ay)^n = 0$  since  $(ay)^n$  is a sum of elements of the form  $(ay_{i_1})(ay_{i_2})^{g_{i_1}} \cdot \dots \cdot (ay_{i_n})^{g_{i_1} g_{i_2} \dots g_{i_{n-1}}} \star g_{i_1} g_{i_2} \dots g_{i_n}$ .

Put now  $B = A/J$ , which is a graded-semisimple algebra, where  $B = \oplus_{i \in I} B e_i$  is a decomposition as a direct sum of graded-simple left  $B$ -modules (see the proof of proposition 2.7). We obviously have that  $\frac{A \star G}{J \star G} \cong \frac{A}{J} \star G = B \star G$ . If we prove that  $B \star G$  is also graded semisimple, then we will have that  $0 = J^{gr}(B \star G) = \frac{J^{gr}(A \star G)}{J \star G}$  and the proof of assertions 2 and 3 will be finished. Indeed we have a decomposition  $B \star G = \oplus_{i \in I} (B \star G) e_i$  and the task reduces to check that  $(B \star G) e_i$  is a graded-simple  $B \star G$ -module, for each  $i \in I$ . Let  $0 \neq x \in (B \star G)_h e_i$  any homogeneous element. Then  $x = \sum_{g \in G} x_g \star g$ , where  $x_g \in B_h e_{g(i)}$  for each  $g \in G$ . Fix a  $\tau \in G$  such that  $x_\tau \neq 0$ . We have that  $0 \neq x_\tau \tau^{-1} \in B_h e_i$  and the graded-simple condition of  $B e_i$  as a graded  $B$ -module gives an element  $y_\tau \in e_i B_{-h}$  such that  $y_\tau x_\tau \tau^{-1} = e_i$ . We now take  $y = y_\tau \star \tau^{-1}$ , so that  $yx = \sum_{g \in G} y_\tau x_g \tau^{-1} \star \tau^{-1} g$ . Note that  $y_\tau x_g \tau^{-1} \in e_i B e_{\tau^{-1} g(i)}$ . By the weakly basic condition of  $A$ , we know that  $e_i A e_j \subseteq J$  (and so  $e_i B e_j = 0$ ) whenever  $j \neq i$ . We then have  $y_\tau x_g \tau^{-1} = 0$ , unless  $\tau^{-1} g(i) = i$ . By the free action of  $G$  on objects, we conclude that  $y_\tau x_g \tau^{-1} \neq 0$  implies that  $\tau = g$ . We then get that  $yx = y_\tau x_\tau \tau^{-1} \star 1 = e_i \star 1$ . It follows that  $(B \star G)x = (B \star G)e_i$ , as desired.

3) By assertions 1 and 2, we only need to check that  $(A \star G) \otimes_A S$  is annihilated by  $J \star G$ . But we have an equality  $J \star G = (A \star G)(J \star 1)$ . It follows that any tensor  $x \otimes z$ , with  $x \in J \star G$  and  $z \in S$ , is zero in  $(A \star G) \otimes_A S$  because  $JS = 0$ . Then we get  $(J \star G)[(A \star G) \otimes_A S] = 0$ .  $\square$

**Definition 11.** Let  $A = \oplus_{h \in H} A_h$  be a graded pseudo-Frobenius algebra and  $G$  be a group acting on  $A$  as graded automorphisms. A graded Nakayama form  $(-, -) : A \times A \rightarrow K$  will be called  $G$ -invariant when  $(a^g, b^g) = (a, b)$ , for all  $a, b \in A$  and all  $g \in G$ .

The following result shows that, in some circumstances, a covering functor preserves the pseudo-Frobenius condition.

**Proposition 4.4.** Let  $A = \oplus_{h \in H} A_h$  be a (split basic) locally bounded graded pseudo-Frobenius algebra, with  $(e_i)_{i \in I}$  as weakly basic distinguished family of orthogonal homogeneous idempotents, and let  $G$  be a group which acts on  $A$  as graded automorphisms which permute the  $e_i$  and which acts freely on objects. If there exists a  $G$ -invariant graded Nakayama form  $(-, -) : A \times A \rightarrow K$ , then  $\Lambda = A/G$  is a (split basic) locally bounded graded pseudo-Frobenius algebra whose graded Nakayama form is induced from  $(-, -)$ .

*Proof.* We put  $\pi := F \circ \iota$ , where  $F$  and  $\iota$  are as in proposition 4.1. We then know that  $\pi$  is surjective on objects and each homogeneous morphism in  $\Lambda$  is a sum of homogeneous morphisms

of the form  $\pi(a)$ , with  $a \in \bigcup_{i,j \in I} e_i A e_j$ . We will put  $\pi(i) = [i]$  and  $\pi(a) = [a]$ , for each  $i \in I$  and homogeneous element  $a \in \bigcup_{i,j \in I} e_i A e_j$ . Note that  $[i]$  and  $[a]$  can be identified with the  $G$ -orbits of  $i$  and  $a$  (see corollary 4.2).

We first check that  $\Lambda$  is weakly basic. The functor  $F$ , which is an equivalence of categories, gives an isomorphism of algebras  $e_i(A \star G)_0 e_i \cong e_{[i]} \Lambda_0 e_{[i]}$ , for each  $i \in I$ . But we have  $e_i(A \star G)_0 e_i = \bigoplus_{g \in G} e_i A_0 e_{g(i)} \star g$ . This algebra is finite dimensional due to the graded locally bounded condition of  $A$  and the fact that  $G$  acts freely on objects. Then all nilpotent elements of  $e_i(A \star G)_0 e_i$  belong to its Jacobson radical. It follows that  $\mathbf{m} := e_i J(A_0) e_i \oplus (\bigoplus_{g \neq 1} e_i A_0 e_{g(i)} \star g)$  is contained in  $J(e_i(A \star G)_0 e_i)$  since, due again to the graded locally bounded condition of  $A$  and the free action of  $G$ , we know that  $\mathbf{m}$  consists of nilpotent elements. Since  $\frac{e_i(A \star G)_0 e_i}{\mathbf{m}} \cong \frac{e_i A_0 e_i}{e_i J(A_0) e_i}$  is a division algebra, we conclude that  $\mathbf{m} = J(e_i(A \star G)_0 e_i)$  and that  $e_{[i]} \Lambda_0 e_{[i]} \cong e_i(A \star G)_0 e_i$  is a local algebra. Moreover, we have that  $\frac{e_{[i]} \Lambda_0 e_{[i]}}{e_{[i]} J(\Lambda_0) e_{[i]}} \cong \frac{e_i(A \star G)_0 e_i}{e_i J((A \star G)_0) e_i} \cong \frac{e_i A_0 e_i}{e_i J(A_0) e_i}$ , so that  $\Lambda$  is split whenever  $A$  is so.

We next prove that  $e_{[i]} \Lambda_h e_{[j]} \subset J^{gr}(\Lambda)$ , whenever  $[i] \neq [j]$  or  $A$  is basic and  $h \neq 0$ . This will give the weakly basic condition of  $\Lambda$  and, in case  $A$  is basic, also its basic condition. But that amounts to prove that  $e_i(A \star G)_h e_j = \bigoplus_{g \in G} [e_i A_h e_{g(j)} \star g] \subset J^{gr}(A \star G)$ , whenever  $[i] \neq [j]$  of  $A$  is basic and  $h \neq 0$ , because the pushdown functor  $F_\lambda : A \star G - Gr \rightarrow \Lambda - Gr$  is an equivalence of categories. The desired fact follows directly from Lemma 4.3 and the definition of weakly basic and basic algebra.

We pass to define the graded Nakayama form for  $\Lambda$ . We will define first graded bilinear forms  $\langle -, - \rangle : e_{[i]} \Lambda e_{[j]} \times e_{[k]} \Lambda e_{[l]} \rightarrow K$ , for all objects  $[i], [j], [k]$  and  $[l]$  of  $\Lambda$ . When  $[j] \neq [k]$  the bilinear form is zero. In case  $[j] = [k]$ , we need to define  $\langle \pi(a), \pi(b) \rangle$  whenever  $a \in \bigoplus_{g, g' \in G} e_{g(i)} A e_{g'(j)}$  and  $b \in \bigoplus_{g, g' \in G} e_{g(j)} A e_{g'(l)}$ . We define  $\langle \pi(a), \pi(b) \rangle$  when  $a, b \in \bigcup_{i, l \in I} e_i A e_l$ , with  $[t(a)] = [i(b)] = [j]$  and then extend by  $K$ -bilinearity to the general case. Indeed we define  $\langle [a], [b] \rangle = (a, b^g)$ , where  $g \in G$  satisfies that  $g(i(b)) = t(a)$ . Note that  $g$  is unique since  $G$  acts freely on objects. We leave to the reader the routine task of checking that  $\langle -, - \rangle : e_{[i]} \Lambda e_{[j]} \times e_{[j]} \Lambda e_{[k]}$  is well-defined. The graded bilinear form  $\langle -, - \rangle : \Lambda \times \Lambda \rightarrow K$  is defined as the 'direct sum' of the just defined graded bilinear forms.

We next check that it satisfies all the conditions of definition 4. We first check condition 2 in that definition. Let  $x, y \in \bigcup_{[i], [j]} e_{[i]} \Lambda e_{[j]}$  be such that  $\langle x, y \rangle \neq 0$ . Then we know that there is  $j \in I$  such that  $t(x) = [j] = i(y)$ . Fix such index  $j \in I$ . Since the functor  $\pi : A \rightarrow \Lambda$  is covering it gives bijections  $\bigoplus_{g \in G} e_{g(i)} A e_j \xrightarrow{\cong} e_{[i]} \Lambda e_{[j]}$  and  $\bigoplus_{g \in G} e_j A e_{g(k)} \xrightarrow{\cong} e_{[j]} \Lambda e_{[k]}$ , for all  $G$ -orbits of indices  $[j]$  and  $[k]$ . We then put  $x = \sum_{g \in G} \pi(a_g)$  and  $y = \sum_{g \in G} \pi(b_g)$  such that  $a_g \in e_{g(i)} A e_j$  and  $b_g \in e_j A e_{g(k)}$ , for all  $g \in G$ . By definition of  $\langle -, - \rangle$ , we then have  $0 \neq \langle x, y \rangle = \sum_{g, g' \in G} (a_g, b_{g'})$ , which implies that there are  $g, g' \in G$  such that  $(a_g, b_{g'}) \neq 0$ . This implies that  $g'(k) = \nu(g(i))$ , where  $\nu$  is the Nakayama permutation associated to  $(-, -)$ . But, due to the  $G$ -invariant condition of  $(-, -)$ , we have that  $\nu(g(i)) = g(\nu(i))$ . This shows that  $[k] = [\nu(i)]$ . It follows that  $\langle e_{[i]} \Lambda, \Lambda e_{[k]} \rangle \neq 0$  implies that  $[k] = [\nu(i)]$ . Therefore assertion 2 of definition 4 holds, and the bijection  $\bar{\nu} : I/G \xrightarrow{\cong} I/G$  maps  $[i] \rightsquigarrow [\nu(i)]$ .

The  $G$ -invariance of  $(-, -)$  also implies that if  $\mathbf{h} : I \rightarrow H$  is the degree function associated to  $(-, -)$ , then  $h(g(i)) = h(i) \forall i \in I$ . As a consequence, the graded bilinear form  $\langle -, - \rangle : e_{[i]} \Lambda \times \Lambda e_{[\nu(i)]} \rightarrow K$  is of degree  $h_i := \mathbf{h}(i)$ , for each  $i \in I$ . Moreover, for each  $j \in I$ , the induced graded bilinear form  $\langle -, - \rangle : e_{[i]} \Lambda e_{[j]} \times e_{[j]} \Lambda e_{[\nu(i)]} \rightarrow K$  is nondegenerate since so is  $(-, -) : e_i A e_{g(j)} \times e_{g(j)} A e_{\nu(i)} \rightarrow K$ , for each  $g \in G$ . Then condition 3 of Definition 4 is satisfied by  $\langle -, - \rangle$ , by taking  $\bar{\mathbf{h}} : I/G \rightarrow H$ ,  $[i] \rightsquigarrow h_i$ , as the degree map.

It remains to check that  $\langle xy, z \rangle = \langle x, yz \rangle$ , for all  $x, y, z \in \Lambda$ . For that, it is not restrictive to assume that  $x = [a]$ ,  $y = [b]$  and  $z = [c]$ , where  $a, b, c$  are homogeneous elements in  $\bigcup_{i, j \in I} e_i A e_j$ . In such a case, note that if one member of the desired equality  $\langle xy, z \rangle = \langle x, yz \rangle$  is nonzero, then  $t(x) = i(y)$  and  $t(y) = i(z)$  or, equivalently,  $[t(a)] = [i(b)]$  and  $[t(b)] = [i(c)]$ . If this holds, then we have

$$\langle xy, z \rangle = \langle [a][b], [c] \rangle = \langle [ab^g], [c] \rangle = (ab^g, c^{g'}),$$

where  $g, g' \in G$  are the elements such that  $g(i(b)) = t(a)$  and  $g'(i(c)) = g(t(b))$ . Note that then  $(g^{-1}g')(i(c)) = t(b)$  and, hence, we also have

$$\langle x, yz \rangle = \langle [a], [b][c] \rangle = \langle [a], [bc^{g^{-1}g'}] \rangle = (a, (bc^{g^{-1}g'})^g) = (a, b^g c^{g'}).$$

The equality  $\langle xy, z \rangle = \langle x, yz \rangle$  follows then from the fact that  $(-, -) : A \times A \longrightarrow K$  is a graded Nakayama form for  $A$ .  $\square$

The following is an easy consequence of last proposition and its proof. We leave the proof to the reader.

**Corollary 4.5.** *Let  $A = \bigoplus_{h \in H} A_h$  be a weakly basic locally bounded graded pseudo-Frobenius algebra and let  $(-, -) : A \times A \longrightarrow K$  be a  $G$ -invariant graded Nakayama form. The following assertions hold:*

1. *If  $\eta : A \longrightarrow A$  is the Nakayama automorphism associated to  $(-, -)$ , then  $\eta \circ g = g \circ \eta$ , for all  $g \in G$*
2. *Let  $\langle -, - \rangle : \Lambda \times \Lambda \longrightarrow K$  be the graded Nakayama form induced from  $(-, -)$  and let  $\bar{\eta} : \Lambda \longrightarrow \Lambda$  be the associated Nakayama automorphism. Then  $\bar{\eta}([a]) = [\eta(a)]$  for each  $a \in \bigcup_{i,j} e_i A e_j$ .*

The following result completes the last proposition by showing how to construct  $G$ -invariant graded Nakayama forms in the split case.

**Corollary 4.6.** *Let  $A = \bigoplus_{h \in H} A_h$  be a split graded pseudo-Frobenius algebra and let  $G$  be a group of graded automorphisms of  $A$  which permute the  $e_i$  and acts freely on objects. There exist an element  $\mathbf{h} = (h_i)_{i \in I} \in \prod_{i \in I} \text{Supp}(e_i \text{Soc}_{gr}(A))$  and basis  $\mathcal{B}_i$  of  $e_i A_{h_i} e_{\nu(i)}$ , for each  $i \in I$ , satisfying the following properties:*

1.  $h_i = h_{g(i)}$ , for all  $i \in I$
2.  $g(\mathcal{B}_i) = \mathcal{B}_{g(i)}$  and  $\mathcal{B}_i$  contains an element of  $e_i \text{Soc}_{gr}(A)$ , for all  $i \in I$

*In such case the graded Nakayama form associated to the pair  $(\mathcal{B}, \mathbf{h})$  (see definition 7) is  $G$ -invariant.*

*Proof.* We fix a subset  $I_0 \subseteq I$  which is a set of representatives of the  $G$ -orbits of objects. Then the assignment  $i \rightsquigarrow [i]$  defines a bijection between  $I_0$  and the set of objects of  $\Lambda = A/G$ . For each  $i \in I_0$ , we fix an  $h_i \in \text{Supp}(e_i \text{Soc}_{gr}(A))$  and a basis  $\mathcal{B}_i$  of  $e_i A_{h_i} e_{\nu(i)}$  containing an element  $w_i \in e_i \text{Soc}_{gr}(A)$ , for each  $i \in I_0$ . Note that  $g(e_i \text{Soc}_{gr}(A)) = e_{g(i)} \text{Soc}_{gr}(A)$  since  $G$  consists of graded automorphisms. It then follows that  $h_i \in \text{Supp}(e_{g(i)} \text{Soc}_{gr}(A))$ . Given  $j \in I$ , the free action of  $G$  on objects implies that there are unique elements  $i \in I_0$  and  $g \in G$  such that  $g(i) = j$ . We then define  $h_j = h_i$  and  $\mathcal{B}_j = g(\mathcal{B}_i)$ , whenever  $j = g(i)$ , with  $i \in I_0$ . Note that  $\mathcal{B}_j$  contains the element  $g(w_i)$  of  $e_j \text{Soc}_{gr}(A)$ . It is now clear that  $\mathbf{h} = (h_j)_{j \in I}$  is in  $\prod_{j \in I} \text{Supp}(e_j \text{Soc}_{gr}(A))$  and that  $\mathcal{B}_j$  is a basis of  $e_j A_{h_j} e_{\nu(j)}$  containing an element of  $e_j \text{Soc}_{gr}(A)$ , for each  $j \in I$ . It is also clear that if  $\mathcal{B} := \bigcup_{j \in I} \mathcal{B}_j$  then  $g(\mathcal{B}) = \mathcal{B}$ , for all  $g \in G$ .

By definition of the graded Nakayama form  $(-, -) : A \times A \longrightarrow K$  associated to  $(\mathcal{B}, \mathbf{h})$  (see definition 7) and the fact that  $w_j = g(w_j) = w_{g(j)}$ , for all  $g \in G$  and  $j \in I$ , we easily conclude that  $(-, -)$  is  $G$ -invariant.  $\square$

We are ready to prove the main result of this section.

**Theorem 4.7.** *Let  $A$  be a split weakly basic locally bounded graded algebra with enough idempotents and let  $G$  be a group of graded automorphisms acting freely on objects. The following assertions are equivalent:*

1.  *$A$  is pseudo-Frobenius;*
2.  *$\Lambda = A/G$  is pseudo-Frobenius.*

*Proof.* 1)  $\implies$  2) By Corollary 4.6, we have a  $G$ -invariant graded Nakayama form  $(-, -) : A \times A \longrightarrow K$ . Now Proposition 4.4 gives this implication.

2)  $\implies$  1) Let  $\langle -, - \rangle : \Lambda \times \Lambda \longrightarrow K$  be a graded Nakayama form for  $\Lambda$  and let  $\pi : A \longrightarrow A/G = \Lambda$  the canonical functor. We shall define a bilinear form  $(-, -) : A \times A \longrightarrow K$  as the 'direct sum' of bilinear forms  $(-, -) : e_i A_h e_j \times e_k A_{h'} e_l \longrightarrow K$ , where  $h, h' \in H$  and  $i, j, k, l \in I$ . By definition, this last bilinear form is zero when  $j \neq k$  and is the composition

$$(-, -) : e_i A_h e_j \times e_j A_{h'} e_l \xrightarrow{\pi \times \pi} e_{[i]} \Lambda_h e_{[j]} \times e_{[j]} \Lambda_{h'} e_{[l]} \xrightarrow{< -, - >} K,$$

when  $j = k$ . Note that this last bilinear form is zero when  $[l] \neq \nu([i])$  or  $h + h' \neq h_{[i]} = \mathbf{h}([i])$ , where  $\nu$  and  $\mathbf{h}$  are the Nakayama permutation and the degree map associated to  $< -, - >$ . We then get that the induced bilinear form  $(-, -) : e_i A e_j \times e_j A e_l$  is either zero, when  $[l] \neq \nu([i])$ , or it is bilinear form of degree  $h_{[i]}$  otherwise. We claim that the resulting bilinear form  $(-, -) : A \times A \rightarrow K$  is a graded Nakayama form for  $A$ , so that  $A$  will be graded pseudo-Frobenius. To prove our claim we first check the following two conditions:

- a) For each  $i \in I$ , there is a unique  $\mu(i) \in I$  such that  $(e_i A, A e_{\mu(i)}) \neq 0$ ;
- b) For each  $j \in I$ , there is a unique  $\mu'(j) \in I$  such that  $(e_{\mu'(j)} A, A e_j) \neq 0$ .

We check a) since b) follows by symmetry. By propositions 3.2 and 3.6, we know that  $e_{[i]} \text{Soc}_{gr}(\Lambda)_h e_{\nu([i])}$  is a one-dimensional  $K$ -vector space whenever  $h \in \text{Supp}(e_{[i]} \text{Soc}_{gr}(\Lambda))$ . Moreover, by the  $G$ -covering condition of the functor  $\pi : A \rightarrow \Lambda$ , we know that we have an isomorphism of  $K$ -vector spaces  $\tilde{\pi} : \bigoplus_{j \in I, \pi(j) = \nu([i])} e_i A_h e_j \xrightarrow{\cong} e_{[i]} \Lambda_h e_{\nu([i])}$ . We claim that  $\tilde{\pi}(e_i \text{Soc}_{gr}(A)_h e_j) \subseteq e_{[i]} \text{Soc}_{gr}(\Lambda)_h e_{\nu([i])}$ , for each  $j \in I$  such that  $\pi(j) = \nu([i])$ . Indeed, if  $w \in e_i \text{Soc}_{gr}(A)_h e_j = \text{Soc}_{gr}(e_i A)_h e_j$ , then  $w \star 1$  is annihilated on the right by  $J \star G = J^{gr}(A \star G)$ , so that  $w \star 1 \in e_i \text{Soc}_{gr}((A \star G)_h e_j)$ . We now view  $w \star 1$  as a morphism  $e_j(A \star G) \rightarrow (A \star G)_h e_j$  in the category  $grproj - A \star G$  of finitely generated projective graded right  $A \star G$ -module. We then use the fact that  $F_\lambda : Gr - A \star G \rightarrow Gr - \Lambda$  is an equivalence of categories since  $\Lambda$  is a weak skeleton of  $A \star G$ . We then get that  $\tilde{\pi}(w \star 1)$  is a morphism  $e_{\nu([i])} \Lambda \rightarrow e_{[i]} \Lambda$  which is the socle of the category of  $grproj - \Lambda$ . That is, it is annihilated by any morphism in the radical of this category, which is easily identified with  $J^{gr}(\Lambda)$ . We then get that  $\tilde{\pi}(w \star 1) \in e_{[i]} \text{Soc}_{gr}(\Lambda)_h e_{\nu([i])}$ , as desired.

The last paragraph shows that we have an induced injective map  $\tilde{\pi} : \bigoplus_{j \in I, \pi(j) = \nu([i])} e_i \text{Soc}_{gr}(A)_h e_j \rightarrow e_{[i]} \text{Soc}_{gr}(\Lambda)_h e_{\nu([i])}$ , whenever  $h \in \text{Supp}(e_{[i]} \text{Soc}_{gr}(\Lambda))$ . We will prove that its domain is nonzero, which will imply that  $\tilde{\pi}$  is bijective since  $\dim(e_{[i]} \text{Soc}_{gr}(\Lambda)_h e_{\nu([i])}) = 1$  (see Proposition 3.6). It will follow also that there is exactly one  $j \in I$  such that  $e_i \text{Soc}_{gr}(A)_h e_j \neq 0$ . For our desired proof, we look at  $[i]$  and  $\nu([i])$  as elements of a set of representatives  $I_0$  of the  $G$ -orbits (see the explicit construction of a weak skeleton). The condition that  $e_{[i]} \text{Soc}_{gr}(\Lambda)_h e_{\nu([i])} \neq 0$  is then equivalent to the condition that  $e_i \text{Soc}_{gr}(A \star G)_h e_j \neq 0$ , for any  $j \in I$  such that  $\pi(j) = \nu([i])$ . Take then  $0 \neq x \in e_i \text{Soc}_{gr}(A \star G)_h e_j$ . We then have  $x = \sum_{g \in G} x_g \star g$ , where  $x_g \in e_i A_h e_{g(j)}$ , for each  $g \in G$ . Since  $x$  is annihilated by  $J^{gr}(A \star G) = J \star G$ , we immediately get that each  $x_g$  is annihilated by  $J = J^{gr}(A)$ , which implies that  $x_g \in e_i \text{Soc}_{gr}(A)_h e_{g(j)}$  and, obviously, there is a  $g \in G$  such that  $x_g \neq 0$ .

We now define  $h_i := \tilde{\mathbf{h}}(i) = \mathbf{h}([i])$ , for each  $i \in I$ , thus obtaining a map  $\tilde{h} : I \rightarrow H$ . The last paragraph gives a map  $\mu : I \rightarrow I$  which assigns to  $i$  the unique  $\mu(i) \in I$  such that  $e_i \text{Soc}_{gr}(A)_{h_i} e_{\mu(i)} \neq 0$ . Note that, by a symmetric argument, we get a map  $\mu' : I \rightarrow I$  which assigns to each  $j$  the unique  $\mu'(j) \in I$  such that  $e_{\mu'(j)} \text{Soc}_{gr}(A)_{h_{\mu'(j)}} e_j \neq 0$ . This implies that  $\mu$  and  $\mu'$  are mutually inverse maps. Note also that, by construction, we have  $(ab, c) = (a, bc)$  whenever  $a, b, c$  are homogeneous elements in  $\bigcup_{i, j \in I} e_i A e_j$ . We then have the following chain of double implications:

$$\begin{aligned} (e_i A, A e_j) \neq 0 &\iff \text{there is a } k \in I \text{ such that } (e_i A e_k, e_k A e_j) \neq 0 \iff \text{there are homogeneous} \\ &\quad \text{elements } a \in e_i A e_k, b \in e_k A e_j \text{ such that } (a, b) \neq 0 \iff \text{there are homogeneous elements} \\ &\quad a \in e_i A e_k, b \in e_k A e_j \text{ such that } (e_i, ab) \neq 0 \iff \text{there is a homogeneous element } w \in e_i A e_j \text{ such} \\ &\quad \text{that } 0 \neq (e_i, w) = \langle e_{[i]}, [w] \rangle \iff \text{there is a homogeneous element } w \text{ of degree } h_i = \mathbf{h}([i]) \text{ such} \\ &\quad \text{that } [w] \in e_{[i]} \text{Soc}_{gr}(\Lambda) e_{[j]}. \end{aligned}$$

We then have  $[j] = \nu([i])$  and the previous paragraphs show that  $w \in \text{Soc}_{gr}(A)_{h_i}$  and then necessarily  $j = \mu(i)$ . Then all conditions of Definition 4 have been checked for  $(-, -) : A \times A \rightarrow K$ , except the fact that the induced bilinear form  $(-, -) : e_i A e_j \times e_j A e_{\mu(i)} \rightarrow K$  is nondegenerate. But this is a consequence of the nondegeneracy of  $\langle -, - \rangle : e_{[i]} \Lambda e_{[j]} \times e_{[j]} \Lambda e_{\nu([i])} \rightarrow K$ . Then  $(-, -)$  is a graded Nakayama form for  $A$ .  $\square$

Although the concept of covering of quivers with relations was initially defined for stable translations quivers (see [18], [8], [3]), the most general version can be found in [9] and [13]. For us a

covering of a quiver with relations  $F : (Q', \rho') \longrightarrow (Q, \rho)$  is what is called a morphism of graphs with relations in [9]. The important fact for us is that if  $F$  is such a covering, then there is a group of automorphisms  $G$  of  $(Q', \rho')$  such that if  $A' = KQ' / \langle \rho' \rangle$  and  $KQ / \langle \rho \rangle$  are the associated algebras with enough idempotents, then  $G$  is an automorphism of  $A'$  which acts freely on objects and we have a covering of (trivially graded)  $K$ -categories  $A' \xrightarrow{\pi} A'/G \xrightarrow{\cong} A$ . Moreover, any quiver with relation  $(Q, \rho)$  admits a *universal covering*  $\tilde{F} : (\tilde{Q}, \tilde{\rho}) \longrightarrow (Q, \rho)$  in the sense that if  $F : (Q', \rho') \longrightarrow (Q, \rho)$  is any covering of quiver with relations, then there is a unique covering of quiver with relations  $F' : (\tilde{Q}, \tilde{\rho}) \longrightarrow (Q', \rho')$  such that  $F \circ F' = \tilde{F}$ . In this setting, it is important to note that if  $A = KQ / \langle \rho \rangle$  is locally finite dimensional, then it is locally bounded if, and only if, so is  $\tilde{A} = K\tilde{Q} / \langle \tilde{\rho} \rangle$  (and so is  $A' = KQ' / \langle \rho' \rangle$ ).

We are now ready to state the following consequence of theorem:

**Corollary 4.8.** *Let  $(Q, \rho)$  be a quiver with relations such that  $A = KQ / \langle \rho \rangle$  is a self-injective finite dimensional algebra. The following assertions hold:*

1. *If  $(\tilde{Q}, \tilde{\rho}) \longrightarrow (Q, \rho)$  is the universal covering, then  $\tilde{A} = K\tilde{Q} / \langle \tilde{\rho} \rangle$  is a pseudo-Frobenius algebra with enough idempotents.*
2. *If  $(Q', \rho') \longrightarrow (Q, \rho)$  is a covering of quivers with relations such that  $A' = KQ' / \langle \rho' \rangle$  is finite dimensional, then  $A'$  is a self-injective algebra.*

*Proof.* Assertion 1 is a direct consequence of theorem 4.7 while assertion 2 follows from this same theorem and from corollary 3.3.  $\square$

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